Banach spaces-Homework 2

Due Date: February 8

1. Let $Y$ be a normed space and $X$ a Banach space. Let $Z \subset Y$ be a dense subset, and $T : Z \to X$ a linear map, continuous at 0. Show that $T$ admits a unique linear extension to $Y$.

2. Let $X$ be a Banach space and $(x_n) \subset X$ be a sequence of linear independent elements. Let $Y_{00} = \bigcup_n Y_n$ be the normed space given by the union of the finite dimensional spaces $Y_n = \text{span}\{x_k : k \leq n\}$. Define the linear map

$$P_n(\sum_{k=1}^m a_k x_k) = \sum_{k=1}^n a_k x_k$$

on $Y_{00}$. Now assume that $\sup_n \|P_n\|$ is finite and show that $(x_n)$ is a basis for the closure $Y$ of $Y_{00}$. (Hint you can show that

$$x = \lim_n P_n(x)$$

holds first for $x \in Y_{00}$, and then for $x \in Y$.)

3. (a) Let $(\Omega, \Sigma, \mu)$ a measure space and $\Sigma' \subset \Sigma$ a finite sub $\sigma$-algebra. Then $\Sigma'$ has atoms $A_1, ..., A_n$ and $|\Sigma'| = 2^n$. Show that the conditional expectation

$$E(f|\Sigma') = \sum_{k=1}^n \frac{\int_{A_k} f d\mu}{\mu(A_k)} 1_{A_k}$$

is contractive on all $L_p$, $1 \leq p \leq \infty$.

(b) Consider $[0,1]$ and the Haar functions defined as follows $h_0 = 1$, $h_1 = \begin{cases} +1 & 0 \leq x \leq 1/2 \\ -1 & 1/2 < x \leq 1 \end{cases}$. Later on they are defined on blocks of cardinality $2^n$ so that

$$h_{2^n+j} = \begin{cases} 1 & \frac{2j}{2^n+1} \leq x \leq \frac{2j+1}{2^n+1} \\ -1 & \frac{2j}{2^n+1} < x \leq \frac{2(j+1)}{2^n+1} \\ 0 & \text{else} \end{cases}$$

for $j = 0, ..., 2^n - 1$. This means $h_{2^n+j}$ is supported on the interval $I^n_j = [j2^{-n}, (j+1)2^{-n}]$. Show that the sequence $(h_j)_{j \geq 0}$ is a basic sequence.
(c) For which values of $1 \leq p \leq \infty$ does $(h_j)$ give a basis in $L_p[0,1]$?

(d) Consider $I : L_p[0,1] \to L_p[0,1]$ given by $I(f)(x) = \int_0^x f(t)dt$. For which values of $1 \leq p \leq \infty$ is the sequence $(I(h_j))$ a basic sequence?