Homework - Hilbert spaces - November

Due date: November 2016.

1) For a Hilbert space $H$, we say that

$$ a \in H \iff \sum_{n=1}^{\infty} \left\| a_n \right\|^2 = \left\| a \right\|^2 \quad \text{holds for some ONB} $$

We also define $b_x^* = \overline{b_x} = (\overline{b_x}, a_f)$

Show that

1. $a, b \in HS \implies b_r(ab) = b_r(ba)$
2. $b \in B(H) \implies b \in HS \implies b_r(ab) = b_r(ba)$

Show that $b_x^*(ux'v) = b_x^*(x') \forall x' \in HS$ and all unitaries $u,v$

Show that $HS(H)$ does not depend on the choice of the basis

2) Let $T_S$ be commutating operators. Define $T_k = \left\{ \begin{align*}
    T_k & = I_{k < 0} = s^0, \quad f > 0 \\
    T_k & = I_{k > 0} = s^0, \quad f < 0
\end{align*} \right.$

Show that

$$ \sum_{k=1}^{\infty} a_{x} \omega^k \cdot s^2 \leq \sup_{|\omega| = 1} \left\| \sum_{k=1}^{\infty} a_{x} \omega^k \cdot s^2 \right\|_{HS(H^2)} $$

$$ \text{If} \quad \left\{ T_k \right\}_{k=1}^{\infty} \quad \left\{ s^0 \right\}_{f>0} \quad \Rightarrow 0 $$

3) Let $u$ be a state with GNS rep $(\mathcal{H}_u, \pi_u, \Omega_u)$

Let $\pi: \Lambda \rightarrow B(H)$ be representation and $\psi$ such that

$$ \psi(a) = (\psi, \pi(a) \psi) \quad \forall a \in \Lambda $$

a) Show that there exists $W \in \mathcal{H}_u \overset{\psi}{\rightarrow} \pi(a) \mathcal{H}_u$ which is an isom. iso.

b) Assume that $V: \Lambda \rightarrow H$ satisfies

$$ [V, \pi(a)\psi] = 0 \quad \forall a \in \Lambda $$

Show that there exists a vector $y \in \mathcal{H}_u$ such that

$$ (\psi, \pi(a) \psi) = (y, \pi(a) s_y) \quad \forall a \in A $$

Now complete the proof of the result in class that for every $y \in \mathcal{H}_u$
there exists $h, k \in \text{Hilb}_u$ such that
$\varphi(a) = (h, \tau \alpha, b)$

c) How can this argument be modified in the separable category?