Hilbert spaces-Math 546

Due date: Wednesday October 12

(1) 3, 5 and 11 in Conway
(2) 2 of the second page in Conway

EXERCISES

1. Prove the uniqueness statement in Proposition 3.4 for the case that $\mathscr{A}$ is abelian.

2. Prove Proposition 3.5.

3. Let $A \in \mathscr{B}(L^2(0,1))$ be defined by $(Af)(t) = tf(t)$. Show that $A \geqslant 0$ and find $A^{1/n}$.

4. Let (X, Ω, μ) be a σ-finite measure space, let $\phi \in L^\infty(X, \Omega, \mu)$, and define M_ϕ as in Theorem II.1.5. Show that $M_\phi \geqslant 0$ if and only if $\phi(x) \geqslant 0$ a.e. $[\mu]$. What is $M_\phi^{1/n}$? If $M_\phi \in \operatorname{Re} \mathscr{B}(\mathscr{H})$, find the positive and negative parts of M_ϕ.

5. Find an example of a positive operator on a Hilbert space that has a nonhermitian square root.

6. If $a \in \operatorname{Re} \mathscr{A}$, show that $|a| \equiv (a^2)^{1/2} = a_+ + a_-$.

7. If $a \in \mathscr{A}_+$, show that $x^* a x \in \mathscr{A}_+$ for every x in $\mathscr{A}$.

8. If $a, b \in \mathscr{A}$, $0 \leqslant a \leqslant b$, and a is invertible, then b is invertible and $b^{-1} \leqslant a^{-1}$.

9. If $a, b \in \operatorname{Re} \mathscr{A}$, $a \leqslant b$, and $ab = ba$, then $f(a) \leqslant f(b)$ for every increasing continuous function f on $\mathbb{R}$.

10. If $a \in \operatorname{Re} \mathscr{A}$ and $\|a\| \leqslant 1$, show that a is the sum of two unitaries. (Hint: First solve this for $\mathscr{A} = \mathbb{C}$.)

11. If $\alpha > 0$, define $f_\alpha\colon (-\alpha^{-1}, \infty) \to \mathbb{R}$ by $f_\alpha(t) = t/(1 + \alpha t) = \alpha^{-1}[1 - (1 + \alpha t)^{-1}]$. Show:

    (a) If $0 \leqslant a \leqslant b$ in $\mathscr{A}$, $f_\alpha(a) \leqslant f_\alpha(b)$ for all $\alpha > 0$;
    (b) $f_\alpha(t) < \min\{t, \alpha^{-1}\}$ for $t > 0$;
    (c) $\lim_{\alpha \to 0} f_\alpha(t) = t$ uniformly on bounded intervals in $[0, \infty)$;
    (d) if $0 \leqslant \alpha \leqslant \beta$, $f_\alpha \leqslant f_\beta$ on $[0, \infty)$;
    (e) $f_\alpha \circ f_\beta = f_{\alpha + \beta}$;
    (f) $\lim_{\alpha \to \infty} \alpha f_\alpha(t) = 1$ uniformly on bounded intervals in $[0, \infty)$.

2. Let $\mathscr{A} = \{f \in C(\text{cl}\,\mathbb{D}): f \text{ is analytic in } \mathbb{D}\}$ and for f in $\mathscr{A}$ define f^* by $f^*(z) = \overline{f(\bar{z})}$. Show that $\mathscr{A}$ is a Banach algebra, $f^* \in \mathscr{A}$ when $f \in \mathscr{A}$, and $\|f^*\| = \|f\|$, but $\mathscr{A}$ is not a C^*-algebra.

3. If $\{\mathscr{A}_i: i \in I\}$ is a collection of C^*-algebras, show that $\oplus_\infty \mathscr{A}_i$ and $\oplus_0 \mathscr{A}_i$ are C^*-algebras.