Hilbert spaces-Math 546

Due date: Wednesday September 21.

i) Let $T_k : X \to X$ be a family of operator such that $\|T_k\| \leq C$, and

$$\lim_k T_k(x) = x$$

for all x in Banach space X. Let $F \subset X$ be a finite dimensional subspace and $\varepsilon > 0$. Show that there is an operator $R : X \to X$ and a k_0 such that

a) $RT_{k_0}(x) = x$ for all $x \in F$

b) $\|R - id\| < \varepsilon$.

(Hint: You may assume that F has a so-called Auerbach basis, i.e. there exists vectors $x_j \in F$, and $x_j^* \in F^*$ such that

$$x_j^*(x_k) = \delta_{kj} \, , \; \|x_k\| = \|x_j^*\| = 1$$

and $x = \sum_j x_j^*(x)x_j$ holds for all $x \in F$. The argument also works with an arbitrary basis, but is messier.)

ii) Let (e_n) be an orthonomal basis in a Hilbert space and H $T : H \to H$ be a compact operator. Show that $\lim_n \|T(e_n)\| = 0$. (Hint a compact operator can be approximated by a matrix with only finitely many non-0 entries.)

iii) b) Let $k \in L_2([0,1]^2)$ and

$$T_k(f)(x) = \int_0^1 k(x,y)f(y)dy \, .$$

Show that T_k is a compact operator. (Hint approximate k be a function

$$\tilde{k}(x,y) = \sum_{i,j} a_{ij} 1_{A_i}(x) 1_{A_j}(y)$$

for disjoint sets (A_i) and (A_j) and estimate

$$\|T_k - T_{\tilde{k}}\| \leq \|k - \tilde{k}\|_{L_2([0,1]^2)} \, . \tag{0.1}$$

If you a problem finding $\tilde{k}$ show at least (0.1) and how to conclude from the fact that thre is $\tilde{k}_\varepsilon$ with $\|k - \tilde{k}_\varepsilon\| \leq \varepsilon$.

iv) Let $H = \ell_2(\mathbb{Z})$ and $V(e_k) = \begin{cases} e_{k+1} & k \neq 0 \\ 0 & k = 0 \end{cases}$. Let $S(e_k) = e_{k+1}$. Is there a continuous path $\gamma : [0,1] \to B(H)$ such that $\gamma(0) = V\text{x}$ and $\gamma(1) = S$?