Hilbert spaces-Math 546

Due date: Wednesday September 14.

i) Let H be a Hilbert space and $A \subset H$ be a non-empty subset. We define

$$A^{\perp} = \{x \in H : \forall a \in A : (x,a) = 0\} .$$

Show that $(A^{\perp})^{\perp}$ is the closure of

$$\mathrm{span}A = \{\sum_{a \in A} \lambda_a a \,:\, \lambda_a \in A, \text{ only a finite number of the } \lambda_a\text{'s are not } 0 \,\} .$$

ii) Let $A = B(H) = L(H)$ be the bounded operators on a Hilbert space. We say that $a \in B(H)$ is positive if

$$(h, ah) \geq 0$$

holds for all $h \in H$. Let X be right A module, i.e. there is a map $m : X \otimes A \to X$ such that

$$m(((x_1+x_2), a)) = m(x_1, a)+m(x_2, b) \quad \text{and} \quad m(x, a+cb) = m(x, a)+m(m(x, c), b) .$$

From now on we write $x.a = m(x, a)$. Let $\phi : \bar{X} \times X \to A$ be a bilinear map such that

i) $\phi(x, y.b) = \phi(x, y)b$,

ii) $\phi(x, y)^* = \phi(y, x)$,

iii) $\sum_{k,j} \phi(x_k, x_j) \geq 0$ for all finite sequence (x_k).

Show that

$$\|x\| = \|\phi(x, x)\|^{1/2}$$

defines a norm on X. Hint: prove the Cauchy-Schwarz inequality

$$\phi(x, y) + \phi(y, x) \leq \phi(x, x) + \phi(y, y) .$$

iii) Find an example of a bounded operator $T : \ell_2 \to \ell_2$ whose range is not closed.