Hilbert spaces-Math 546

Due date: Wednesday September 7.

A matrix $a \in \mathbb{M}_n(\mathbb{C})$ is called *positive* (semidefinite) (in short $a \geq 0$) if

$$(h, ah) \geq 0$$

holds for all $h \geq 0$. The (Froebenius) norm of a matrix is given by

$$\|a\| = \sup_{h\neq 0} \frac{\|ah\|}{\|h\|} .$$

Here and in the following $\|h\| = (\sum_k |h_k|^2)^{1/2}$.

i) Let $a \in \mathbb{M}_m(\mathbb{C})$ be a selfadjoint matrix.

a) Show that

$$4Re(h, ak) = (h+k, a(h+k)) - ((h-k), a(h-k)) .$$

b) Find a formula for

$$8(h, ak)$$

using $+, -, i, -i$ (polarization formula), and correct the 8 to a 4 if necessary.

b) Let a be a seljadjoint matrix. Show that

$$\|a\| = \sup_{\|h\|\leq 1} |(h, ah)| .$$

c) Show that there exists $\lambda \in \{\|a\|, -\|a\|\}$ and $h \neq 0$ such that

$$ah = \lambda h .$$

(Hint: Use the following fact: Given a vector h of norm 1, we can use the orthogonal projection and write an arbitrary vector $h' = \lambda h + k$ such that $\|h'\|^2 = |\lambda|^2 + \|k\|^2$.)

d) Show that a selfadjoint matrix is diagonalizable, more precisely there is an orthonormal basis x_1,...,x_n and such that

$$a(x_i) = \lambda_i x_i .$$

e) Show that that a matrix a is positive if and only if a is selfajoint and all the eigenvalues are positive.

ii) Let $[a,b]$ be a interval and μ a measure such that $\mu([x,x+\varepsilon)) > 0$ for all $x \in [a,b)$ and some $\varepsilon > 0$. Apply the Gram-Schmidt procedure to find an orthonormal basis. Why does the procedure not stop? Show also that there exists linearly independent polynomials such that

$$\int_a^b P_k P_j \mu \; = \; \delta_{kj} \int P_k^2 d\mu$$

and

$$P_{n+1}(x) \; = \; x P_n(x) + a_n P_n(x) + b_n P_{n-1}(x) \; .$$

Hint: Apply Gram-Schmidt to the monomials.