Hilbert spaces-Math 568

Due date: Wednesday September 13.

i) Let $w : [a, b] \to \mathbb{R}_+$ be an integrable function such that for some interval $[c, d]$ the Lebesgue measure $m$ satisfies

$$m\{x \in [c, d] : w(x) > 0\} = d - c .$$

We define the scalar product on the space of real continuous functions

$$(f, g) = \int_a^b f(x)g(x)w(x)dm(x) .$$

Show that is a sequence $(p_n)$ of polynomials with real coefficients such that

i) $p_n$ has degree $n$ and the leading coefficient is 1, i.e. $p_n(x) = x^n + a_{n,1}x^{n-1} + \cdots + a_{n,0}$.

ii) $(p_n, p_k) = 0$ for $k \neq k$.

iii) There exist coefficients $a_n$ and $b_n$ such that

$$p_n(x) = xp_{n-1}(x) + a_np_{n-1}(x) + b_np_{n-1}(x) .$$

Hint: One can construct $p_n$ inductively and find a formula for $a_n$ and $b_n$ in terms of the $p_{n-1}$'s and $xp_{n-1}$.

ii) Let $H$ be a Hilbert space and $A \subset H$ be a non-empty subset. We define

$$A^\perp = \{ x \in H : \forall a \in A : (x, a) = 0 \} .$$

Show that $(A^\perp)^\perp$ is the closure of

$$\text{span} A = \{ \sum_{a \in A} \lambda_a a : \lambda_a \in A, \text{ only a finite number of the } \lambda_a \text{'s are not 0 } \} .$$

iii) a) Let $a$ be a $n \times n$ matrix. Show that $B = a^*a$ is a selfadjoint matrix with positive entries and hence (why) can be written as $a^*a = u^*D\lambda u$ where $u$ is a unitary and $D\lambda(e_k) = \lambda_k$ is a diagonal operator with positive entries. Define

$$b = u^*D\sqrt{\lambda}u .$$

Show that there is a unitary $w$ such that

$$a = wb .$$

(Hint: First define $w$ on the image of $b$ and then extend it).
b) If not already done complete the proof of

\[ \| (x, y) \| \leq \|(x, x)\|^{1/2}\|(y, y)\|^{1/2}. \]