Hilbert spaces-Math 568

Due date: Wednesday September 13.

i) Let $w : [a,b] \to \mathbb{R}_+$ be an integrable function such that for some interval $[c,d]$ the Lebesgues measure m safisfies

$$m\{x \in [c,d] : w(x) > 0\} \;=\; d - c \, .$$

We define the scalar product on the space of real continuous functions

$$(f,g) \;=\; \int_a^b f(x)g(x)w(x)dm(x) \, .$$

Show that is a sequence (p_n) of polynomials with real coefficients such that

  i) p_n has degree n and the leading coefficient is 1, i.e. $p_n(x) = x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,0}$.

  ii) $(p_n, p_k) = 0$ for $k \neq k$.

  iii) There exist coefficients a_n and b_n such that

$$p_n(x) \;=\; xp_{n-1}(x) + a_n p_{n-1}(x) + b_n p_{n-1}(x) \, .$$

Hint: One can construct p_n inductively and find a formula for a_n and b_n in terms of the p_{n-1}'s and xp_{n-1}.

ii) Let H be a Hilbert space and $A \subset H$ be a non-empty subset. We define

$$A^\perp \;=\; \{x \in H : \forall a \in A \, : \, (x,a) = 0\} \, .$$

Show that $(A^\perp)^\perp$ is the closure of

$$\text{span}A \;=\; \{\sum_{a \in A} \lambda_a a \; : \; \lambda_a \in A, \text{ only a finite number of the } \lambda_a\text{'s are not } 0 \; \} \, .$$

iii) a) Let a be a $n \times n$ matrix. Show that $B = a^*a$ is a selfadjoint matrix with positive entries and hence (why) can be written as $a^*a = u^*D_\lambda u$ where u is a unitary and $D_\lambda(e_k) = \lambda_k$ is a diagonal operator with positive entries. Define

$$b \;=\; u^* D_{\sqrt{\lambda}} u \, .$$

Show that there is a unitary w such that

$$a \;=\; wb \, .$$

(Hint: First define w on the image of b and then extend it).

b) If not already done complete the proof of

$$\|(x,y)\| \;\leq\; \|(x,x)\|^{1/2}\|(y,y)\|^{\frac{1}{2}}\;.$$