Hilbert spaces-Math 568

Due date: Wednesday September 13.

i) Let \( w : [a, b] \rightarrow \mathbb{R}_+ \) be an integrable function such that for some interval \([c, d]\) the Lebesgue measure \( m \) satisfies
\[
m\{x \in [c, d] : w(x) > 0\} = d - c.
\]

We define the scalar product on the space of real continuous functions
\[
(f, g) = \int_{a}^{b} f(x)g(x)w(x)dm(x).
\]

Show that \((p_n)\) is a sequence of polynomials with real coefficients such that
i) \( p_n \) has degree \( n \) and the leading coefficient is 1, i.e. \( p_n(x) = x^n + a_{n, n-1}x^{n-1} + \cdots + a_{n, 0} \).

ii) \( (p_n, p_k) = 0 \) for \( k \neq k \).

iii) There exist coefficients \( a_n \) and \( b_n \) such that
\[
p_n(x) = xp_{n-1}(x) + a_n p_{n-1}(x) + b_n p_{n-1}(x).
\]

Hint: One can construct \( p_n \) inductively and find a formula for \( a_n \) and \( b_n \) in terms of the \( p_{n-1} \)'s and \( xp_{n-1} \).

ii) Let \( H \) be a Hilbert space and \( A \subset H \) be a non-empty subset. We define
\[
A^\perp = \{x \in H : \forall a \in A : (x, a) = 0\}.
\]

Show that \((A^\perp)^\perp\) is the closure of
\[
\text{span} A = \{\sum_{a \in A} \lambda_a a : \lambda_a \in A, \text{ only a finite number of the } \lambda_a \text{'s are not 0} \}.
\]

iii) a) Let \( a \) be a \( n \times n \) matrix. Show that \( B = a^*a \) is a selfadjoint matrix with positive entries and hence (why) can be written as \( a^*a = u^* D_\lambda u \) where \( u \) is a unitary and \( D_\lambda(e_k) = \lambda_k \) is a diagonal operator with positive entries. Define
\[
b = u^* D_{\sqrt{\lambda}} u.
\]

Show that there is a unitary \( w \) such that
\[
a = wb.
\]

(Hint: First define \( w \) on the image of \( b \) and then extend it).
b) If not already done complete the proof of
\[ \|(x, y)\| \leq \|(x, x)\|^{1/2} \|(y, y)\|^{1/2}. \]