Practice solutions

PB1: Let $f \in L_1$ and g continuous such that $f|_I = g|_I$, I an open interval. We want to show that the non-tangential limits coincide. We consider $h = f - g$ and want to show that the limits are 0 on I. Let $x \in I$ and $(x-\varepsilon, x+\varepsilon) \subset I$. We may assume $x = 0$. Let $z_n = r_n e^{i\theta_n}$. Since $\lim_n z_n = 1$, we may assume $|t_n| \leq \varepsilon/2$ for $n \geq n_0$-look at cone. This means for $|t| \geq \varepsilon$ that

$$\cos(t - t_n) \leq \cos(\varepsilon - \varepsilon/2) \; = \; 1 - \delta \; .$$

Look at graph for cosine. However, this implies

$$P_r(t_n - t) \leq \frac{1 - r_n^2}{(1-r_n)^2 + 2r_n\delta} \; .$$

The later limit converges to 0, and hence for $n \geq n_\gamma$ we have

$$\sup_{|t| \geq \varepsilon} P_r(t_n - t) \; \leq \; \gamma$$

and

$$|P(f-g)(r_n e^{it_n})| \; \leq \; \gamma \|f - g\|_1 \; \leq \; \gamma(\|f\|_1 + \|g\|_\infty) \; .$$

Since γ is arbitrary, we deduce the assertion.

PB2 We want to understand the non-tangential limits for a simple function. According to Problem 1, we only have to consider the jump from 0 to 1 in a neighborhood. Therefore it suffices to consider the case where $f(x) = 0$ for $-\pi \leq x \leq 0$ and 1 for $0 \leq x \leq \pi$, and our limit point is 0. But the Poisson measure is symmetric, i.e. $P_r(t) = P_r(-t)$ and hence

$$\int P_r(t) f(t) \frac{dt}{2\pi} \; = \; \frac{1}{2} \int P_t(t) \frac{dt}{2\pi} \; = \; \frac{1}{2} \; .$$

Thus, using the argument from Problem 1, we get that

$$\lim_{z_n \in N_\alpha} P[g](z_n) \; = \; 1/2$$

for ever function g which is 0 on $(-\varepsilon, 0)$ and 1 on $(0, \varepsilon)$ for some $\varepsilon > 0$.

PB3 We want to assume that g is continuous up to a finite number of points and that left and right limits exist !!!!!!

For every point of continuity, we can use PB1 and have convergence. For a point of discontinuity x_0, and $\delta > 0$, we choose a $\varepsilon > 0$ such that

$$|g(t) - g^{\pm}(t_0)| < \varepsilon \; .$$

Let h be the function with values $g^-(t_0)$ for $t < t_0$ and $g^+(t_0)$. By PB 3, we know that

$$\lim_{z \in e^{it_0} N_\alpha} P[g1_{(t_0-\delta,t_0+\delta)}] \;=\; \frac{g^-(t_0)+g^+(t_0)}{2} \,.$$

However,

$$\sup_z |P[g1_{(t_0-\delta,t_0+\delta)} - f1_{(t_0-\delta,t_0+\delta)}](z)| \;\le\; \|g1_{(t_0-\delta,t_0+\delta)} - f1_{(t_0-\delta,t_0+\delta)}\|_\infty \;\le\; \varepsilon \,.$$

The good news here is that limit does not depend on δ. Moreover, the solution of problem one shows that for $\varepsilon' > 0$ and given δ_n we can choose r_n such that for $z = re^{it}$ in the cone and $r \;\ge\; r_n$ we have

$$|P(g1_{(t_0-\delta_n,t_0+\delta_n)} - f1_{(t_0-\delta,t_0+\delta)}| \;\le\; \varepsilon' \|P(g1_{(t_0-\delta_n,t_0+\delta_n)} - f1_{(t_0-\delta,t_0+\delta)}\|_1 \;\le\; \varepsilon'\varepsilon_n \,.$$

We may simply do this $\varepsilon' = 1$ and deduce the assertion by sending ε_n to 0.

PB 4 It is very easy to see that for $f = 1_{[-1/2,1/2]}$ we have

$$Mf(x) \;\ge\; \frac{1}{2x}$$

by using the interval $[0, x]$ or $[x, 0]$. This

$$\|Mf\|_{L_1(\mathbb{R})} \;\ge\; 2\int_1^T \frac{1}{2x} dx \;=\; \ln T$$

holds for all $T > 0$ and hence converges to ∞. Now we show that there is no constant c such that

$$\|Mg\|_{L_1([-1/2,1/2])} \;\le\; c\|g\|_{L_1([-1/2,1/2])}$$

holds for all for positive $g \in L_1([-1/2, 1/2])$. Indeed, we define $f_T(x) = Tf(Tx)$. Then a change of variable chose that

$$\int_{-1/2}^{1/2} f_T(x)dx \;=\; \int_{-1/2}^{1/2} f(Tx)Tdx \;=\; \int_{-1/2}^{1/2} f(y)dy \;=\; 1 \,.$$

On the other hand it is easy to see that $Mf_T(x) = TMf(Tx)$ and hence

$$\|Mf_T\|_1 \;\ge\; \ln T \,.$$

Let us now construct a positive function F such that

$$\|MF\|_1 \;=\; \infty$$

and $\|F\|_1 = 1$. Indeed, let f_T from above. Then we may use

$$F \;=\; \sum_{n=1}^\infty \frac{1}{cn^2} f_{e^{n^4}} \,,$$

where $c = \sum_{n=1}^{\infty} 1/n^2$ is finite and

$$MF \;\geq\; \frac{1}{cn^2} \ln e^{n^4} \;\geq\; c^{-1} n^2 \,.$$

And what the heck is the connection to the Poisson integral?
Let us resume: we have found a positive function $F \in L_1([-\pi, \pi])$ such that $\|F\|_1 = 1$ and $\|MF\|_1 = \infty$. Now we want to show that radial maximal function can not be in L_1. We need the following

Lemma 0.1. *Let* $0 < r < 1$ *and* $y = (1 - r)$. *Then*

$$\frac{1}{2y} \int_{-y}^{y} f(t)dt \;\leq\; \pi \int P_r(t) f(t) dt \,.$$

Proof. We know that $1 - \cos(t) \leq t^2/2$ and hence $t \leq (1 - r)$ implies

$$\begin{aligned} P_r(t) &= \frac{1 - r^2}{1 + r^2 - 2r\cos(t)} \\ &\geq \frac{1 - r^2}{(1 - r)^2 + rt^2} \;\geq\; \frac{(1 - r)(1 + r)}{(1 + r)(1 - r)^2} \qquad \geq \frac{1}{1 - r} \;=\; \frac{1}{\pi} \frac{2\pi}{2(1 - r)} \,. \end{aligned}$$

Thus for positive f we have

$$\int P_r(t) f(t) \;\geq\; \frac{1}{\pi} \int_{-y}^{y} f(t) \frac{dt}{2y} \,.$$

This proves the Lemma. ∎

We deduce that

$$Mf(t) \;\leq\; \pi \sup_{0 \leq r \leq 1} P[f](re^{it})$$

holds for positive f, and hence the radial maximal function does not belong to $L_1([-\pi, \pi])$ for our f from above.