Due date: November 14

(1) p 276 1
(2) p 276 4
(3) p 276 6
(4) p 276 9
The fact that \((h)\) holds in every simply connected region has the following consequence (which can also be proved by quite elementary means):

13.12 Theorem If \(f \in H(\Omega)\), where \(\Omega\) is any open set in the plane, and if \(f\) has no zero in \(\Omega\), then \(\log |f|\) is harmonic in \(\Omega\).

Proof: To every disc \(D \subset \Omega\) there corresponds a function \(g \in H(D)\) such that \(f = e^g\) in \(D\). If \(u = \Re g\), then \(u\) is harmonic in \(D\), and \(|f| = e^u\). Thus \(\log |f|\) is harmonic in every disc in \(\Omega\), and this gives the desired conclusion. //

Exercises

1. Prove that every meromorphic function on \(S^2\) is rational.

2. Let \(\Omega = \{z: |z| < 1\}\) and \(|2z - 1| > 1\), and suppose \(f \in H(\Omega)\).
   - (a) Must there exist a sequence of polynomials \(P_n\) such that \(P_n \to f\) uniformly on compact subsets of \(\Omega\)?
   - (b) Must there exist such a sequence which converges to \(f\) uniformly in \(\Omega\)?
   - (c) Is the answer to \((b)\) changed if we require more of \(f\), namely, that \(f\) be holomorphic in some open set which contains the closure of \(\Omega\)?

3. Is there a sequence of polynomials \(P_n\) such that \(P_n(0) = 1\) for \(n = 1, 2, 3, \ldots\), but \(P_n(z) \to 0\) for every \(z \neq 0\), as \(n \to \infty\)?

4. Is there a sequence of polynomials \(P_n\) such that

\[
\lim_{n \to \infty} P_n(z) = \begin{cases} 
1 & \text{if } \Im z > 0, \\
0 & \text{if } z \text{ is real}, \\
-1 & \text{if } \Im z < 0.
\end{cases}
\]

5. For \(n = 1, 2, 3, \ldots\), let \(\Delta_n\) be a closed disc in \(U\), and let \(L_n\) be an arc (a homeomorphic image of \([0, 1]\)) in \(U - \Delta_n\) which intersects every radius of \(U\). There are polynomials \(P_n\) which are very small on \(\Delta_n\) and more or less arbitrary on \(L_n\). Show that \(\{\Delta_n\}, \{L_n\}\), and \(\{P_n\}\) can be so chosen that the series \(f = \sum P_n\) defines a function \(f \in H(U)\) which has no radial limit at any point of \(T\). In other words, for no real \(\theta\) does \(\lim_{r \to 1} f(re^{i\theta})\) exist.

6. Here is another construction of such a function. Let \(\{n_k\}\) be a sequence of integers such that \(n_1 > 1\) and \(n_{k+1} > 2kn_k\). Define

\[
h(z) = \sum_{k=1}^{\infty} 5^k z^{n_k}.
\]

Prove that the series converges if \(|z| < 1\) and prove that there is a constant \(c > 0\) such that \(|h(z)| > c \cdot 5^n\) for all \(z\) with \(|z| = 1 - (1/n_k)\). [Hint: For such \(z\) the \(m\)th term in the series defining \(h(z)\) is much larger than the sum of all the others.]

Hence \(h\) has no finite radial limits.

Prove also that \(h\) must have infinitely many zeros in \(U\). (Compare with Exercise 15, Chap. 12.)

In fact, prove that to every complex number \(z\) there correspond infinitely many \(z \in U\) at which \(h(z) = z\).

7. Show that in Theorem 13.9 we need not assume that \(A\) intersects each component of \(S^2 - \Omega\). It is enough to assume that the closure of \(A\) intersects each component of \(S^2 - \Omega\).

8. Prove the Mittag-Leffler theorem for the case in which \(\Omega\) is the whole plane, by a direct argument which makes no appeal to Runge's theorem.

9. Suppose \(\Omega\) is a simply connected region, \(f \in H(\Omega)\), \(f\) has no zero in \(\Omega\), and \(n\) is a positive integer. Prove that there exists a \(g \in H(\Omega)\) such that \(g^n = f\).