**Definition:** \( (f, D) \) is called a branched element if \( D \) is an open disc and \( f \in h(D) \).

**Definition:** \( (f_1, D_1) \sim (f_2, D_2) \) if
\[
f_1|_{D_1 \cap D_2} = f_2|_{D_1 \cap D_2} \quad \text{and} \quad D_1 \cap D_2 \neq \emptyset
\]

Warning: \( \sim \) is not transitive.

**Lemma:** \( (f_0, D_0) \sim (f_1, D_1) \sim (f_2, D_2) \) and
\[
D_1 \cap D_2 \cap D_3 \neq \emptyset \Rightarrow (f_0, D_0) \sim (f_2, D_2)
\]

**Proof:** \( f_1|_{D_1 \cap D_3} \) and \( f_2|_{D_2 \cap D_3} \) are analytic.
\( e(f_1-f_0) \subset D_1 \cap D_2 \cap D_3 \) have no critical points.
\( \Rightarrow \) \( f_1 = f_2 \) on \( \text{int} \left( D_1 \cap D_2 \cap D_3 \right) \).

**Lemma:** Let \( g \) be a curve, and \( \tau(0) = x \in D \).

\( (f, D) \) admits at most one extension along curve \( g \).

**Proof:**

Let \( t = t_0 \) be a maximal point such that \( g(t_0) = f(t_0) \).
Then \( t_0 < s \) such that \( t_0 \in D_{t_0} \) and \( s \in D_{t_0} \).

For \( s \leq t \leq s_0 \), \( D_t \cup D_s \neq \emptyset \Rightarrow (f, D_t) \sim (f, D_s) \)
\[\Rightarrow (f, D_{t_0}) \sim (f, D_{s_0}) \]
and we can go a little further.

**Lemma:** Let \( \gamma \) be a homotopy of curves between \( \alpha \) and \( \beta \).

Such that \( (f, D) \) can be continued along \( \gamma \) to get
\[\Rightarrow (f, D_{t_0}) \sim (f, D_{s_0}) \]
Proof \(\gamma_t \in \bigcup_{i=1}^{n} A_i\) be contained in an open set and \(\varepsilon = \text{dist}(\gamma_t, \partial(UA))\).

By uniform continuity, \(\text{dist}(\gamma_t, \varepsilon_t) < \varepsilon/2\).

Thus \(\text{Lem} \gamma_t \in \Omega \cap A_t \sim (\pi_t, D_t) \sim (\pi_t, \partial D_t)\).

Use a finite cover of \([0,1]\) by compact sets.

**Lemma:** \(\Omega\) simply connected. Then two curves connecting \((\alpha, \beta)\) are part of homotopy.

**Proof**

\[\gamma_0 \rightarrow \gamma_2\] simply connected \[\rightarrow\] homeomorphism.

But on \(\bigcup_{i=1}^{n} A_i\) we have connecting union.

Push forward with \(\mathcal{F}\).

**Theorem (Hurwitz)** Let \(\Omega\) be simply connected.

\(\alpha \in \Omega\) \(D \subseteq \Omega\) with center \(d\). Assume that for every \(\beta\) we can find a curve \(\alpha \rightarrow \beta\) with analytic continuation (uniquely determined).

\[\exists \mathcal{F} \subseteq H(\mathbb{C}) \text{ s.t. } \mathcal{F}|_D = \mathcal{F}\]

**Proof**

\[\Gamma \rightarrow \Gamma_2\] all about \((D, \beta) \sim (D, \beta')\).

This means \((D, \beta)\) is uniquely determined.
\[ D_p \cup D_p' = 0 \Rightarrow g_p \big|_{D_p \cup D_p'} = g_{p}' \big|_{D_p \cup D_p'} \]

\[ g(z) = \begin{cases} g_\beta(z) & z \in D_\beta \\
\end{cases} \]
is well defined.

Note: Homotopy theorem fails for non-simply connected?
Upper half plane and $SL(2)$

Recall that we have a map

$$\psi : \mathcal{C}(\mathbb{R}) \to \mathcal{H}(\mathbb{C}^2)$$

$$\psi(\frac{a+bi}{c+di}) = \frac{a+bi}{c+di}$$

**Lemma:** $a,b,c,d \in \mathbb{Z}; \quad ad-bc = 1$

1. $\psi_{(a,b,c,d)}(1) = \psi(1)$

**Proof:** $\psi_{(a,b,c,d)}(1) = \frac{(ad-bc)1+ac}{(ac+bd)+bd}$

$$\frac{ad-bc}{ac+bd} \neq 0 \iff \psi(1) \neq 0$$

**Consider** $\mathbb{C}(x) = \mathbb{C}[x] = \{ \frac{a}{b} \mid a,b \in \mathbb{C} \}$

$$\mathbb{C}(x) = \left\{ \frac{a}{b} \mid a,b \in \mathbb{C}, \text{gcd}(a,b) = 1 \right\}$$

**Lemma:** $\psi_{(a,b,c,d)}(x) = \psi(x)$

1. $\psi_{(a,b,c,d)}(1) = \psi(1)$

**Proof:** $\psi_{(a,b,c,d)}(1) = \frac{(ad-bc)1+ac}{(ac+bd)+bd}$

$$\psi_{(a,b,c,d)}(x) = \psi(x)$$

Proof of (v) (Sketch)

$$\Sigma = \bigcup_{\psi(1) \in \Sigma} \psi(1) \subset \mathbb{H}^+$$

Choose $\psi(1) \in \Sigma$ such that $\text{gcd}(a,b) = 1$ and $a > 0$

$$\psi(1) = \frac{a+bi}{c+di}$$

Then $\Sigma = \{ \psi(1) \} = \bigcup_{\psi(1) \in \Sigma} \psi(1)$

$$\psi_{(a,b,c,d)}(1) = \frac{(ad-bc)1+ac}{(ac+bd)+bd}$$

For $z \in \mathbb{C}$

$$z \in \mathbb{H}^+ \iff \psi_{(a,b,c,d)}(z) = \psi(z)$$
Theorem: The exterior of $\mathbb{D}(0,1)$, denoted $\mathbb{D}(0,1)^*$, is connected.

Proof:

1. $\partial \mathbb{D}(0,1) = \{z \in \mathbb{C} : |z| = 1\}$ is compact and connected.
2. $\mathbb{D}(0,1)^* = \mathbb{C} \setminus \{\partial \mathbb{D}(0,1)\}$.
3. $\partial \mathbb{D}(0,1)$ is a closed set.
4. $\mathbb{D}(0,1)^*$ is the complement of a compact set in $\mathbb{C}$.

Lemmas:

1. $\mathbb{D}(0,1)^*$ is open.
2. $\partial \mathbb{D}(0,1)^* = \partial \mathbb{D}(0,1)$.

By the Riemann removability theorem, $\mathbb{D}(0,1)^*$ is connected.
\[ |g^m(x)| = \frac{M}{R} \]

\[ |g^m(x)| \leq \frac{1}{2} \left( \int |\beta \alpha(x)| \right) \leq 2 \epsilon \]

\[ |g(x)| \leq \frac{1}{2} \left( \int |\beta \alpha(x)| \right) \leq 2 \epsilon \]

Hence, \[ |g(x)| \leq \frac{1}{2} \left( \int |\beta \alpha(x)| \right) \leq 2 \epsilon \]

\[ R \to \infty \]

\[ n \to \infty \]

\[ \sum \]

\[ S \subseteq \partial (\Omega) \]

\[ \sum \]

\[ \text{conv} \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]
Theorem: \( f: C \to C \) is a branched function such that 
\( f(z) \) omits two points. Then \( f \) is constant.

Proof: Wlog, we may assume that \( z = 0, \beta = 1 \) are not contained in range \( (\text{Moebius transform}) \).

Let \( \lambda: \mathbb{T}^+ \to \mathbb{R} \) be the analytic map constructed in the previous theorem.

Let \( D_1 \subset \mathbb{R} \) be a disc. Then there exists a region \( \nu_\lambda \subset \mathbb{T}^+ \) such that 
\( \lambda(\nu_\lambda) = D_1 \).

In fact, \( \lambda \) maps \( \nu_\lambda \) to \( D_1 \) such that \( \lambda(\nu_\lambda) = D_1 \).

Let \( D_1 \cap D_2 \neq \emptyset \). Choose \( V_2 \). \( V_1 \cap V_2 \neq \emptyset \).

\[ \implies (\nu_1, D_2) \sim (\nu_2, D_2) \]

Choose \( z_0 = 0 \). \( f(z_0) \in D_0 \in \nu \).

\[ \exists \psi_\lambda \quad \lambda(\psi_\lambda(z_0)) = z \quad z \in \nu_\lambda \]

Then \( (\psi_0, A_0) \) can be continued along \( \gamma \).

\[ \exists \gamma: \mathbb{C} \to \mathbb{T}^+ \]

Then we compose \( \gamma \) with \( \varphi: \mathbb{T}^+ \to \mathbb{D} \) \( \varphi(z) = \frac{z-i}{z+i} \).

\( \gamma \circ \varphi \) is mono-bounded hence constant.

\[ \implies \gamma \circ \varphi \text{ is constant} \implies f \text{ locally constant} \implies f \text{ constant} \]
Questions and problems for final

Questions

i) What can you say about zero’d of an analytic function in a region? What
principles allow you to construct functions which are 0 a given set. what
does this have to do with Blaschke products?

ii) Same question for meromorphic functions in a region, ...

iii) How can you recognize a simple connected domain by looking at the ring
$H(\Omega)$?

iv) For which $\Omega$ is $H(\Omega)$ principal (i.e. every finitely generated idea is single
generated)? Did you see this structure before?

v) When can you say something on the boundary behaviour of the function
given by the Riemann mapping theorem? To which extend can the Riemann
mapping theorem be ‘unique’?

Problems:

(1) 22 page on page 252
(2) 9 page 264
(3) 11 page 264
(4) 10 page 277
(5) 11 page 277
(6) 3 page 293
(7) 9 page 294
(8) 15 page 294
(9) 1 page 352
(10) 8 page 333
Problem 22. Let \( f_n \subset L_1 \) be uniformly integrable. Then \( \text{Re}(f_n) \) and \( \text{Im}(f_n) \) are also uniformly integrable. We show that \((f_n)\) is uniformly integrable iff \((|f_n|)\) is uniformly integrable. Indeed, one can show that \((f_n)\) are not uniformly integrable iff there exists a non-orthogonal sequence \( E_k \) of sets and a subsequence \( f_{n_k} \) such that
\[
\lim_k \lambda(E_k) = 0 \quad \text{and} \quad \inf_k \left| \int_{E_k} f_{n_k} d\lambda \right| > 0.
\]
By replacing \( E_k \) by \( E_k^+ = E_k \cap \{ f_{n+k} > 0 \} \), we deduce that \( f_n^+ \) is also not uniformly integrable. Accepting this criterion therefor shows \((f_n)\) not uniformly integrable implies \((|f_n|)\) not uniformly integrable. Similarly, \((|f_n|)\) not uniformly integrable show that \( f_n \) is not uniformly integrable because
\[
\varepsilon < \int_{E_k} |f_{n_k}| \leq \max \{ \int_{E_k^+} f_{n_k}, \int_{E_k^-} f_{n_k} \}.
\]
Thus \((|f_n|)\) not uniformly integrable shows that \((f_n)\) is not uniformly integrable using this criterion.

Now it is easy to conclude: Since \((f_n)\) uniformly integrable implies \((|f_n|)\) uniformly integrable, we find a \( \delta > 0 \) such that \( |E| < \delta \) implies
\[
\sup_n \int_{E} |f_n| \leq 1.
\]
We apply this to open balls \( B(x, r) \) of Lebesgue measure \(< \delta \) and find a finite
\[
[-\pi, \pi] \subset \bigcup_{j=1}^m B(x_j, r).
\]
This implies
\[
\int |f_n| d\lambda \leq \sum_{j=1}^m \int_{B(x_j, r)} |f_n| d\lambda \leq m
\]
for every \( n \). Hence a uniformly integrable sequence is automatically uniformly bounded in \( L_1 \).

Now assume that
\[
\sup_r \|u_r\|_1 = \infty.
\]
Using the Poisson semigroup \( P_t(u_s) = u_{s+t} \) we see that \( g(r) = \|u_r\| \) is increasing. Hence there exists a subsequence \( r_n \) such that
\[
\lim_n \|u_{r_n}\|_1 = \infty.
\]
Since $u_{r_n}$ is uniformly integrable this contradicts the first part of our proof. Thus
\[ \sup_r \| u_r \| < \infty. \]

Now we can apply the theorem in the book.

**Proof of criterion**  
Assume $(f_n)$ uniformly integrable then certainly no such sequence can exist. Now assume $(f_n)$ not uniformly integrable and $\varepsilon > 0$ given which satisfies the assumptions. Let $\eta_k = 2^{-k}$. Let us define $G_N = \max_{1 \leq k \leq N} |f_k|$. Then there exits a set $E_1$ of measure $\leq 1/2$ and $n_1$ such that
\[ |\int_{E_1} f_N| \geq \varepsilon. \]

There exists $\delta_2 \leq \eta_2$ such that
\[ \sup_{|F| < \delta_2} |\int_F G_N| \leq \frac{\varepsilon}{4}. \]

There exists a set $f_n$ and $|F| \leq \delta_2$ such that
\[ \varepsilon \leq |\int_F f_n|. \]

But then $n > N$. We may write $F = F \cap E_1 \cap F \cap E_1^c$. Then
\[ |\int_{F \cap E_1} f_n| \leq \frac{\varepsilon}{4} \]

and hence
\[ |\int_{E_1} f_N| \geq \varepsilon - \frac{1}{4} \varepsilon. \]

So we define $n_1 = N$, $E_1(2) = E_1 - F$ and $E_2(2) = F$. The we get a sequence
\begin{align*}
E_1(j) &= E_1 - (E_2(2) \cup \cdots E_j(j)) \\
E_2(j) &= E_2(2) - (E_3(3) \cup \cdots E_j(j)) \\
& \vdots \\
E_j(j) &= \cdots
\end{align*}

such that
\begin{align*}
|\int_{E_1(j)} f_{n_1}| &\geq \varepsilon - \sum_{k=1}^{j} 4^{-j} \varepsilon \\
|\int_{E_2(j)} f_{n_2}| &\geq \varepsilon - \sum_{k=2}^{j} 4^{-j} \varepsilon \\
& \vdots \\
|\int_{E_j(j)} f_{n_j}| &\geq \varepsilon.
\end{align*}
Then \( E_1 = \lim_j E_1(j), \ E_2 = \lim_j E_2(j), \ldots \ E_k = \lim_j E_k(j) \) satisfy the assertion.

**Maximum Modulus Principle, Problem 9,10.** Without loss of generality assume that \( 0 < \alpha < 1 \). Choose \( \beta \) such that \( 0 < \alpha < \beta < 1 \). One possible auxiliary function is, for each \( \epsilon > 0 \),

\[ h(z) = \exp(-\epsilon(z+1)^\beta) \]

Prove that \( |f(\epsilon y) h(\epsilon y)| \leq 1 \) for all real \( y \). Show that for \( |z| > R \) for some \( R > 0 \), \( |f(z) h(z)| \leq 1 \). Apply Maximum Modulus Principle on the inside of the half disk of radius \( R > 0 \) in the right half plane to conclude that \( |fh| \leq 1 \) on the closure of \( \Pi \). For each \( z \in \Pi \), let \( \epsilon \to 0 \).

The function \( e^z \) gives an example where this theorem is false when \( \alpha = 1 \).

For the general case, let \( \phi \) be the composition of \( z^{\pi/\delta} \) with a rotation so that \( f \circ \phi^{-1} \) is a function on \( \Pi \). This function must satisfy the inequality stated in the problem, and it produces an inequality which \( f \) must satisfy so that, by the first part of the problem, \( |f| \leq 1 \) in the angular region.

A detailed solution to this problem and its successor is included in the attached file.

**Maximum Modulus Principle, Problem 11.** If \( \Omega \) is the plane, the conclusion holds by Liouville’s theorem.

Let \( B \) be the bound on the boundary and \( M \) be the bound on \( M \). Let \( n > 0 \) be an integer.

Let us assume that \( 0 \) belongs to the boundary of \( \Omega \) and \( \varepsilon > 0 \). Then we can find \( \delta > 0 \) such \( |z| \leq \delta \) implies \( |f(z)| \leq (1 + \varepsilon)B \). We replace \( \Omega \) by \( \Omega_\delta = \Omega \setminus B(\delta) \) and \( \Gamma \) by \( (1 + \varepsilon)\Gamma \). Now we may assume that \( \Omega^c \) contains an open neighborhood of \( 0 \).

Let \( V \) be the disk centered at \( 0 \) of radius \( R > 0 \) large enough so that \( z_0 \in V \).

The boundary of the component of \( V \cap \Omega_\delta \) containing \( z_0 \) consists of points in the boundary of \( V \) (contained in \( \Omega \)) and points in the boundary of \( \Omega_\delta \). For a boundary point of \( \Omega_\delta \cap \partial V \), we have \( |f^n(z)/z| \leq R^{-1}M^\delta \). For the other boundary , we have \( |f^n(z)/z| \leq \delta^{-1}B^n \). Choose \( R \) large enough so that \( M^n/R < \delta((1 + \varepsilon)B)^n \). Apply the Maximum Modulus Principle and hence

\[ |f(z_0)| \leq (1 + \varepsilon)B|z_0|^{1/n} \]

We send \( n \to \infty \), and then \( \varepsilon \to 0 \).

**Simply Connected Domains, Problem 10.** One direction is easy: if \( e^g = f \), then \( e^{g/n} \) is a holomorphic \( n \)th root of \( f \).

For the converse, fix \( z_0 \in \Omega \) and let \( k \geq 0 \) be the order of zero of \( f \) at \( z_0 \). Since \( f \) is not
identically zero, $k$ is well-defined. We show that $k = 0$ by means of contradiction. \nAssume $k > 0$ and let $g \in H(\Omega)$ be such that $g^{2k} = f$. Since we are in a field, $f(z_0) = 0$ implies that $g(z_0) = 0$. It follows that $f$ has order of zero $2k$ at $z_0$, which contradicts.
If follows that $f'/f \in H(\Omega)$. If this function has an antiderivative in $\Omega$, then multiplying an appropriate constant produces a holomorphic logarithm of $f$, and the proof is completed. Choose a point $\alpha$ in the region. For any $z \in \Omega$, choose a path connecting $\alpha$ and $z$ and integrate $f'/f$ over this path. This defines a function on $\Omega$, which may not be well-defined. But if it is well-defined, it is an antiderivative of $f'/f$, and it is well-defined if and only if the line integral of $f'/f$ is zero over all closed path in $\Omega$, which we now show. Let $\gamma$ be a closed path in $\Omega$. We have, for $g$ an $n$th root of $f$,
\[
\text{Ind}_{f \circ \gamma}(0) = \int_{\gamma} \frac{f'}{f} = n \int_{\gamma} \frac{g'}{g} = n \text{Ind}_{g \circ \gamma}(0)
\]
The index of a point with respect to a closed path is integer valued, and the only integer that is a multiple of all positive integers is zero. This finishes the proof.

**Approximation by Rational Functions, Problem 11.** The proof, as noted in the problem, depends on a problem in chapter on holomorphic functions, which states: if a uniformly bounded sequence $(f_n)$ of holomorphic functions converges on a region pointwise to $f$, then the convergence is uniform on every compact subset. To prove this, first note that Montel’s theorem gives a subsequence which converges uniformly on compact subsets. This limit function is $f$, and uniform convergence implies that $f \in H(\Omega)$. Now fix $z_0 \in \Omega$ and choose $V \subseteq U \subseteq \Omega$ be disks centered at $z_0$ of radius $R$ and $2R$ respectively whose closures are contained in the region. Let $\gamma$ be a closed path going around the boundary of $U$ once. Let $M > 0$ be such that $|f_n| \leq M$ on $\Omega$. Then on $V$, we have $|f_n(z)/z - z_0| \leq M/R$ for all $n$. The dominated convergence theorem applied to the Cauchy formula of $f_n - f$ shows that the convergence is uniform on $V$ (In fact, the result follows from the proof of the dominated convergence theorem:the bound function is the same for each $z$ and the pointwise convergence on $\gamma$ is independent of $z$.) This finishes the proof.
We turn to the present problem. As in the hint, for each open disk we wish to find a disk in it on which \( \{ f_n \} \) is uniformly bounded. Once this is done, the union of the collection of all such smaller disks is an open dense set on which, by the previous problem, \( f \) is holomorphic and the convergence is uniform on compact subsets. Choose \( V \) an open disk in \( \Omega \) and define \( \phi = \sup |f_n| \) on \( \Omega \).

The function \( \phi \) is lower semicontinuous so that the set \( B_n = \{ z \in V : |\phi(z)| > n \} \) is open. If \( B_n \) is not dense in \( V \) for some \( n \), the proof is finished. On the other hand, if all \( B_n \) are dense, their intersection is also dense by Baire’s theorem; in particular, there is \( z \in V \) such that \( |\phi(z)| = \infty \). This contradicts pointwise convergence of \( f_n \).

The proof is completed.

**Conformal Mapping, Problem 3.** The following proof depends on a problem in the chapter on Maximum Modulus Principle which states that any entire function which maps the unit circle into itself is of the form \( f(z) = \alpha z^n \) where \( |\alpha| = 1 \). To see this first note that if \( f \) does not reach zero in the unit disk, the maximum modulus principle applied to \( f \) and \( 1/f \) implies that \( |f| = 1 \) on the unit disk and thus on the plane. By Open mapping, \( f \) is constant.

If \( f \) is not constant, consider the function \( f(z)f(1/z) \). This function is holomorphic on the punctured plane \( \mathbb{C} \setminus \{0\} \) and is 1 on the unit circle and thus on the punctured plane (In the proof of Schwarz’s reflection principle we showed that \( z \mapsto f(\overline{z}) \) is holomorphic; use the chain rule. In fact, a reflection principle can be proved for an open set in the unit disk where \( |f| \to 1 \) whenever \( z \) approaches the unit circle \( T \) and the resulting relation is \( f(1/\overline{z}) = 1/F(\overline{z}) \). This proves both assertions.) In particular, \( f(z) \neq 0 \) for \( z \neq 0 \).

If the order of zero of \( f \) at \( z = 0 \) is \( m \), the function \( f(z)/z^m \) is an entire function which sends the unit circle into itself and which does not reach a zero in the unit disk. It is constant by the above discussion. The proof is therefore finished.

The present problem is easily solved with this lemma: First note that from the above remark on reflection principle the given function has a zero at \( \alpha_n \) if and only if it has a pole at \( 1/\overline{\alpha_n} \). The function is rational and so has finitely many zeros and poles. Thus dividing the function by the linear fractional maps \( \phi_{\alpha_n} \) (and powers of \( z \)) produces an entire function of the type in the lemma, noting that each \( \phi_{\alpha_n} \) has modulus 1 on the unit circle. The proof is then completed.

**Conformal Mapping, Problem 9.** For the first part, compose \( e^{(i\pi/2)z} \) with \( z - 1/z + 1 \). The second part follows from the first by taking the real part of the inverse of the function constructed, using the principle branch of the logarithm.
For part (c), we need a lemma, which is Problem 10 in Conformal Mapping: if $f$ and $g$ are holomorphic mappings of $U$ into $\Omega$, $f$ is one-to-one and $f(0) = g(0)$, then $g(D(0, r)) \subseteq f(D(0, r))$. The proof of this is easy: The function $f^{-1} \circ g$ maps $U$ into itself, 0 to 0. By Schwartz’s Lemma, $|f^{-1} \circ g(z)| \leq |z|$.

Let $f$ be the inverse of the function constructed in the first part, again using the principal branch of the logarithm. If $V = D(0, r)$, the point at which $|f(z)|$ is maximal for $z$ in the closure of $V$ is a boundary point of the image and is therefore an image of the circle of radius $r$. For $z = r$, $f(r) = \frac{2\pi}{2}\log\frac{1+r}{1-r}$. I think this maximizes $|f|$ on the closure of $V$. By the lemma, the desired inequality holds.

For the last part, as in the first part construct a one-to-one conformal mapping from the given strip to the unit disk. Call the inverse of this function $f$. Let $\delta = h(\alpha + i\beta)$ and let $\phi_\delta = z - \delta \bar{z}$, $\psi = \phi_\delta \circ f^{-1} \circ h \circ f$ is a one-to-one conformal mapping of the unit disk onto itself, which maps 0 to 0. Apply Schwartz’s lemma to this function and its inverse to show that $|\psi'(0)| = 1$. Compute $|h'(\alpha + i\beta)|$ from this identity. The computation is not very long and is omitted here.

**Conformal Mapping, Problem 15.** The map $h : z \mapsto \frac{z-1}{z+1}$ maps the right half plane injectively onto the unit disk, and it maps 1 to 0. Precomposing $f_n$ with this function produces a sequence of uniformly bounded sequence of functions in $U$. Denote it by $g_n = h \circ f_n$. By Montel’s theorem, $(g_n)$ admits a subsequence which converges uniformly on compact subsets of $U$. Postcompose $h^{-1}$ with this sequence gives a subsequence of $(f_n)$ which converges on compact subsets of the unit disk. The condition $f(0) = 1$ for all $f$ in the family is not needed. However, if $|f(0)| \leq 1$ for all $f \in \mathcal{F}$, then the family is locally bounded.

**Problem 8 page 333** We are given a compact set $K$ on the real line and

$$f(z) = \int \frac{1}{t-z} dt$$

on $\Omega \setminus K$. Let us calculate the real and imaginary part.

$$f(x + it) = \int \frac{(t-x) + iy}{(1-x)^2 + y^2} dt = \int \frac{t-x}{(t-x)^2 + y^2} dt + iy \int \frac{1}{(t-x)^2 + y^2} dt.$$ 

We note that

$$|(x + iy)((t-x) + iy)| = |x(t-x) - y^2 + iyt| \leq |t(t-x)| + |(x-t)|^2 + y^2 + 2y^2t^2.$$ 

Thus for $|x| \geq 2t$ we get

$$|(x + iy)((t-x) + iy)| \leq 10((t-x)^2 + y^2).$$
Let us consider \((z_n)\) which converge to \(\infty\) so that the real part converges to \(\infty\). Then we may apply the dominated convergence theorem and deduce that

\[
\lim_n |z_n f(z_n)| \leq \left| \int_K \lim_n \frac{z_n - t}{z_n - t} + \frac{t}{z_n - t} dt \right| = \lambda(K).
\]

If \(x_n\) remains bounded but \(y_n\) goes to 0, we may still apply the dominated convergence theorem because \(x_n - t\) also remains bounded, thus the same calculation applies. This means \(\lim_{z \to \infty} zf(z) = \lambda(K)\).

The real part need not be bounded. Take \(K = [0, 1] \) and \(x_k = -1/k\), \(y = 0\) then we find

\[
\int (t - 1/k)^{-1} dt \geq \int_{1/k}^{1} 1/tdt = \ln(k).
\]

For the imaginary part we find

\[
|Im(f(z))| \sim |y| \int \min((x - t)^{-2}, y^{-2}) dt = |y| \int_{(x-t)^2 \geq y^2} (x - t)^{-2} + \frac{1}{|y|} \int_{t \in K, |x-t| \leq y} dt \leq 2|y||y|^{-1} + \frac{2|y|}{|y|} \leq 4.
\]

Integrating over a large disc, we can perform Fubini and apply Cauchy's formula and find

\[
\int f(\gamma) \frac{d\gamma}{2\pi i} = \lambda(K).
\]

We have a holomorphic square root iff \(\Omega\) is null-homotopic, and that is not the case if \(K\) is not empty.

The existence of non-trivial bounded constants is not so easy, but answered in problem 10: If \(\lambda(K)\) has measure 0, then \(K\) is removable, i.e. there are no non-constant holomorphic functions on \(K^c\). Also \([-1, 1]\) is not removable. So it depends,.....

No compact connected set with two points is removable (answer no nonconstant bounded holomorphic functions).

Finally if \(f\) is a constant, the \(f\) has to be 0, because \(|z||f(z)|\) is bounded. Thus \(f\) is constant iff \(\lambda(K) = 0\). It seems that \(f\) is a bounded holomorphic function in any case, and so we have completely answered the question about existence of nonconstant bounded functions for \(K\) as part of the real line.
(1) Give a very brief answer for the following questions.

(a) Let $h \in H^\infty$ such that $h^*(e^{it}) = 1$ on a non-trivial interval. What is $h(0)$? \textbf{Solution:} $h(0) = 0$, according a theorem about almost everywhere convergence and a clever use of the Mittag-Leffler theorem.

(b) Describe how to construct a meromorphic function on $D$ which has zero’s in $0, 1, 2$ and poles in $i/2$ and $-i/2$. We take a product for $0$ we take

$$f(z) = \prod (z - z_k) \prod (z - w_k)^{-1}.$$ 

A better solution is given by the Blaschke product for the $0$’s.

(c) Can you construct a meromorphic function on $D$ which has zero’s in $z_n = i - i/n^2$ and poles in the sequence $q_k = 1 - 1/k^2$? Yes, the infinite product

$$\prod_{k=0}^{\infty} (z - z_k)$$

makes sense. Then we use $f(z) = h(z)/g(z)$, where $g$ is the Blaschke product $q_k$ with simple multiplicity.
(2) Let \( \Omega \) be a domain and \( L \) be a line or a circle so that \( \Omega = \Omega_1 \cup (\Omega \cap L) \cup \Omega_2 \) and \( \Omega_1 \) and \( \Omega_2 \) are open. Show that a function \( f \) in \( C(\Omega) \) such that both restriction \( f|_{\Omega_1} \in H(\Omega_1) \), \( f|_{\Omega_2} \in H(\Omega_2) \) are analytic, are analytic. (Hint: You may assume that \( L \) is a line and then apply Morera’s theorem). \textbf{Sol.:} Using a fractional transformation we may replace a circle by a line. Hence it is enough to prove it for a line. Now we have to consider a triangle in \( \Omega \). Such a triangle may not intersect the line at all, and then the converse of Morera’s theorem applies. In the other cases the line can connect two edges, and then we split the triangle in triangle and a trapezoid. By continuity we can replace the triangle by a triangle inside \( \Omega_1 \) and a closed path inside \( \Omega \). By holomorphy these tow integrals are 0, and by uniform on compact sets continuity total integral is smaller than \( \varepsilon \). The same applies if the triangle uses one part of \( L \). Thus Morera’s theorem implies the assertion. \( \blacksquare \)
(3) Let $\Pi^+$ be the upper halfplane, $f \in H(\Pi^+)$ with continuous extension to the boundary and $0 < \alpha < 1$ such that

\[
|f(x + iR)| \leq e^{cR^\alpha} \\
\lim_{|x| \to \infty} \sup_{0 \leq r \leq R} |f(x + ir)| = 0.
\]

Show that

\[
|f(z)| \leq \sup_{x \in \mathbb{R}} |f(x)|.
\]

(Hint: Three line Lemma)

**Solution:** Let $\Gamma$ be the bound on the real axis. Define $f_R(z) = f(z/R)$. For $R$ large enough we apply the three line Lemma

\[
|f(z_0)| = \Gamma^{1-y/R} e^{cR^\alpha y/R}.
\]

The second term converges to 1 for $R$ to $\infty$ and we are done.
(4) We want to understand the boundary behaviour of function \( f \in H^\infty(\Pi^+) \).

Consider \( \phi(x) = \frac{x-i}{x+i} \).

(a) Verify that in this case the Riemann mapping theorem provides a function \( \Phi : \Pi^+ \to D \) which extends continuously to \( \mathbb{R} \).

(b) Find a measure on \( \mathbb{R} \) such that
\[
\int f(\phi(x))d\mu(x) = \int_{\partial D} f(e^{it})\frac{dt}{2\pi}.
\]

(c) Let \( g \in H^\infty(\pi^+) \). Show that the limits
\[
\lim_{y \to 0} g(x + iy)
\]
exists almost everywhere (wrt Lebesque measure).

**Sol.:** We note that for \( x = a + ib \) we have
\[
|\frac{x - i}{x + i}|^2 = \frac{a^2 + (b-1)^2}{a^2 + (b+1)^2} \leq 1
\]
iff \((b-1)^2 \leq (b+1)^2 \) iff \(-2b \leq 2b \) iff \( b \geq 0 \). This means that \( \phi \) is a Riemann map, and Riemann maps are unique up to automorphism of \( U \). We see that \( \phi \) extends continuously to the boundary with \( \phi(\infty) = 1 \). For the second problem, we consider
\[
t = \log \frac{x - i}{x + i} = \log(1 + \frac{2i}{x + i})
\]
Then
\[
\frac{dt}{dx} = -2i(x + i)^{-2} \frac{x + i}{x - i} = -2i \frac{1}{(x + i)(x - i)} = \frac{2i}{x^2 + 1}.
\]
Using the absolute value in the transformation formula we get
\[
\int f(\phi(x)) \frac{2}{x^2 + 1} dx = \int_{-\pi}^{\pi} f(e^{it})dt.
\]
This means our measure is absolute continuous wrt to the Lebesgue measure.

For the third problem, we consider a fixed \( x_0 \) and \( e^{it_0} = \frac{x_0 - i}{x_0 + i} \) and
\[
g(t) = \phi(x_0 + it) = \frac{x_0 + i(t-1)}{x_0 + i(t+1)}.
\]
Then
\[
g'(t) = i\frac{(x_0 + i(t+1)) - (x_0 + i(t-1))}{(x_0 + it + 1)^2}
\]
\[
= \frac{2}{(x_0 + it)^2 - i^2}.
\]
Then \( g'(0) = \frac{2}{(x_0+i)} = \frac{2(x_0-i)}{x_0^2+1} \). Note that

\[
\frac{g(x_0)}{x_0} = \frac{x_0 - i}{x_0 + i} = \frac{(x_0 - i)^2}{x_0^2 + 1}.
\]

Hence the vectors are not orthogonal and we can apply non-tangential limits for the curve \( \gamma \) hits the boundary in angle different from the perpendicular angle \( e^{i(t_0-\pi)} \). Thus the theorem about angular convergence on the circle implies almost everywhere convergence for function \( g : \Pi^+ \to \mathbb{C} \) such that \( g|_{\mathbb{R}} \in L_1(\mathbb{R}, \frac{dx}{1+x^2}) \). In particular for bounded functions.
(5) Let \( \phi_n(z) = z^n \) and \( \Phi_n(f) = f \circ \phi_n \) be the corresponding action on \( H(\Pi^+) \).

We want to understand the fixpoint ring \( H_\Gamma(\Pi^+) = \{ f \in H(\Pi^+) | f \circ \phi_n = f \} \).

(This correspond to the holomorphic functions on \( \Pi^+/\Gamma \)).

(a) Let \( S = \{ z | Im(z) > 0, 0 < Re(z) < 1/2 \} \) and \( h : S \rightarrow \Pi^+ \) a Riemann mapping. Why does \( h \) extends to the boundary? What is \( h([0, 1/2]) \).

(b) Assume \( h([0, 1/2]) = [0, 1/2] \). Construct a function \( \lambda : \Pi^+ \rightarrow \Pi^+ \) such that \( \lambda \circ \phi_n = \lambda \).

(c) Show that there exists a ring homomorphism \( \sigma : H(\Pi^+) \rightarrow H_\Gamma(\Pi^+) \).

(d) Does \( \sigma \) extend to an isomorphism? What can be said?

**Sol.:** \( h \) extends to the boundary because every point in \( S \) is simple. Thus \( h \) extends continuously. For \(-1 \leq x \leq 0\), we may define
\[
\lambda(-x + iy) = \lambda(x - iy).
\]

By reflection principle and the lemma we proved before we know that \( \lambda \) is holomorphic on \(-1/2 \leq Re(z) \leq 1/2\). We repeat the procedure and get a \( n\mathbb{Z} \) periodic function \( \lambda : \Pi^+ \rightarrow \Pi \) such that \( \lambda \circ \phi_n = \lambda \). Note that \( h([0, 1/2]) \) is a compact connected set, an hence \( h([0, 1/2]) = \lambda(\mathbb{R}) \) is an interval. Of course, we may assume that \( h([0, 1/2]) = [0, 1/2] \).

Our ring homomorphism is given by \( \sigma(f) = f \circ \lambda \). Then we see that \( \sigma(f)|_S \) is exactly the homomorphism given by the Riemann mapping theorem. Thus \( \sigma \) is certainly injective. Let us now consider a function \( g \in H^+ \) such that \( g \circ \phi_n = g \). Then \( g \) is uniquely determined on \( \{ -1/2 \leq Re(z) \leq 1/2 \} \). An element in \( \sigma(H(\pi^+)) \) satisfies
\[
f(x + iy) = f(\lambda(x + iy)) = f(\ldots)
\]