Homework 1 - Math 541

Due date: Monday Jan 28

(1) Define for a vector $x \in \mathbb{R}^n$ define

$$\|x\| = \inf_{x = y + z} \|y\|_1 + \|z\|_2,$$

where $\|x\|_p = (\sum_{k=1}^{n} |x_k|^p)^{1/p}$. Calculate $\|x\|$ for $x_j = 1/j$.

(2) Let $X = \ell_{\infty}(\mathbb{N})$ be equipped with

$$\|x\| = \sup_n |x_n|$$

(a) a) Show that $c_0(\mathbb{N}) = \{ x | \lim_n x_n = 0 \}$ is closed.

(b) b) Show that there exists an elements $\phi \in \ell_{\infty}^*$ such that $\phi \neq 0$ and $\phi|_{c_0} = 0$.

(c) You may use the Hahn-Banach theorem to show that $c_0^\perp = \ell_1$.

(3) Let $\phi \in \ell_{\infty}^*$ be a functional. Let $T_n : X \to Y^*$ be a sequence of linear maps such that $\|T_n\| \leq 1$ for all $n \in \mathbb{N}$. Show that

$$T_\phi(x)(y) = \phi((T_n(x))(y))_n$$

defines a linear bounded map form $T : X \to Y^*$. Assuming that $\phi(1) = 1$. What do you have to assume so that $T_\phi$ does not depend on $\phi$?