1. Motivation

The instructor of this course is Marius Junge, Atlgeld 363. He will be back on Friday—and then we agree that homework will be turned in every Wednesday, starting January 26. Further details on grading, exam—see Friday.

**Why functional analysis?**

Functional analysis has been developed as common tools in different aspects of Analysis: Linear PDE’s, Banach space Theory, Operator algebras, Convexity, and more recently Quantum Information Theory.

For many researchers Functional Analysis can be described as infinite dimensional Linear Algebra.

In modern research the tools and methods from Functional Analysis serve as a guideline for a more abstract approach to problems in Analysis (with some additional algebraic structure).

**Plan:**

1. Basics in Banach spaces
2. Locally convex topological spaces and the Hahn-Banach Theorem.
4. Duality and uniform convexity in Banach spaces
5. Fancy applications of Baire’s category theorem: open mapping theorem and closed graph theorem
6. Some applications to vector measures.
7. Compact operators adn Fredholm alternative
8. Spectral Theorem on Hilbert spaces

2. Banach spaces

**Definition 2.1.** A normed space is given by a vector space $V$ (over $K = \mathbb{R}$ or $K = \mathbb{C}$) and a function $\| \| : V \to [0, \infty)$ satisfying the following conditions

- i) $\| x \| = 0 \iff x = 0$,
- ii) $\| \lambda x \| = |\lambda|\| x \|$, 
- iii) $\| x + y \| \leq \| x \| + \| y \|$, 

for all $x, y \in V$, $\lambda \in K$. The associated metric on $(V, \| \|)$ is defined by

$$d_\| (x, y) = \| x - y \|.$$
Remark 2.2. $+: V \times V \to V$ given by $+(x, y) = x + y$ and $\cdot: K \times V \to V$ given by $\cdot(\lambda, x) = \lambda x$ are continuous. Moreover, $\|\|: V \to [0, \infty)$ is continuous.

In the following we will mostly consider real vector spaces.

Definition 2.3. A Banach space is a normed vector space such that $(V, d_{\|\|})$ is complete.

Example 2.4. (1) On $V = \mathbb{R}^n$ we define

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$$

and $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$. Then $(\mathbb{R}^n, \|\|_p)$ is Banach space (see below for the triangle inequality).

(2) $\ell_p = \{(x_n) : \sum_{n} |x_n|^p < \infty\}$ is a Banach space with respect to

$$\|(x_n)\|_p = \left(\sum_{n=1}^\infty \|x_n\|^p\right)^{\frac{1}{p}}.$$

(3) If $\|\|$ is a norm on $\mathbb{R}^n$, then $(\mathbb{R}^n, \|\|)$ is a Banach space.

(4) $(C[0,1], \|\|_1)$ where

$$\|f\|_1 = \int_0^1 |f(s)|ds$$

is a normed space, but not a Banach space.

Lemma 2.5. A normed space is complete if every absolutely convergent series is actually converging.

Proof. An absolutely converging series is given by $x_n \in X$ such that

$$\sum_n \|x_n\|$$

is finite. Convergence here means the convergence of

$$S_n = \sum_{k=1}^n x_k$$

Of course, for an absolutely convergent series $S_n$ is Cauchy and hence, in a complete (metric) space it is converging. Conversely assume that $(y_n)$ is Cauchy. Passing to a subsequence, we may assume $\|y_n - y_{n+1}\| \leq 2^{1-n}$. Then we define $x_0 = y_0$
and \( x_n = y_n - y_{n-1} \). The series \( \sum_n \|x_n\| \) is absolutely convergent, thanks to the geometric series. Thus our assumptions implies that
\[
y_n = y_0 + \sum_{j=1}^{n} y_j - y_{j-1} = S_n
\]
is convergent. But for a Cauchy sequence convergence of subsequence implies convergence.

**Proposition 2.6.** Let \( X \) be a normed space and \( Y \) be a Banach space. We define \( L(X,Y) \) as the space of map \( T : X \to Y \) which are linear, i.e.
\[
T(x + \lambda y) = T(x) + \lambda T(y).
\]
and continuous. The norm on \( L(X,Y) \) is given by
\[
\|T\|_{op} = \sup_{\|x\| \leq 1} \|T(x)\|.
\]
Then \( L(X,Y) \) is a Banach space.

**Lemma 2.7.** For a linear map \( T : X \to Y \) the following are equivalent

i) \( T \) is continuous;

ii) \( T \) is continuous at 0;

iii) \( T \) is bounded, i.e. \( \|T\| = \sup_{\|x\| \leq 1} \|Tx\| \) is finite.

iv) \( T \) is Lipschitz

**Proof.** The only non-trivial part is \( ii) \Rightarrow iii \). However \( T^{-1}(B_Y) \) contains a neighborhood of 0, i.e. there exists a \( r > 0 \) such that
\[
\|x\| < r \Rightarrow \|Tx\| < 1.
\]
This implies \( \|T\| \leq 1/r \).

**Proof of Proposition.** We observe that \( \| \|_{op} \) is a norm. We only check the triangle inequality. Indeed,
\[
\|T + S\|_{op} = \sup_{\|x\| \leq 1} \|(T + S)(x)\| = \sup_{\|x\| \leq 1} \|T(x) + S(x)\| \leq \sup_{\|x\| \leq 1} \|T(x)\| + \|S(x)\| \\
\leq \|T\|_{op} + \|S\|_{op}.
\]
Finally we have to show that \( L(X,Y) \) is complete. Let \( (T_n) \) be a Cauchy sequence of linear maps. For fixed \( x \in X \), we have
\[
\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\| \|x\|.
\]
Thus \((T_n(x))\) is Cauchy and we may define
\[
T(x) = \lim_n T_n(x) .
\]

Then we have
\[
T(x + \lambda y) = \lim_n T_n(x + \lambda y) = \lim_n T_n(x) + \alpha T_n(y) = T(x) + \lambda T(y) .
\]

Thus \(T\) is linear. Let us show that
\[
\text{op}\ (2.1) \quad \lim_n \|T - T_n\|_{\text{op}} = 0 .
\]

Indeed, let \(x \in X\) with \(\|x\| \leq 1\). Then we have
\[
\|T(x) - T_n(x)\| = \|\lim_m T_m(x) - T_n(x)\| \leq \limsup_m \|T_m(x) - T_n(x)\|
\leq \sup_{m \geq n} \|T_m - T_n\| \|x\| \leq \sup_{m \geq n} \|T_m - T_n\| .
\]

In particular \(\|T\|_{\text{op}} \leq \|T - T_1\|_{\text{op}} + \|T_1\|_{\text{op}}\) is finite and \(T\) is continuous. Moreover, \(\lim_n d(T, T_n) = 0\) implies that \(\lim_n T_n = T\). \(\blacksquare\)

**Corollary 2.8.** Let \(X\) be a normed space. Then \(X^* = L(X, \mathbb{R})\) is a Banach space. Moreover, \(X^{**} = L(X, \mathbb{R})\) is a Banach space.

**Definition and Remark 2.9.** Let \(\iota : X \rightarrow X^{**}\) be the linear map given by \(\iota(x)(x^*) = x^*(x)\). Then
\[
\|\iota(x)\| \leq \|x\| .
\]

Indeed, the Hahn-Banach theorem (proved later in this course) shows that \(\|\iota(x)\| = \|x\|\). A Banach space \(X\) is called reflexive if \(\iota(X) = X^{**}\), i.e. \(\iota\) is surjective. All finite dimensional spaces are reflexive.

**Definition and Remark 2.10.** Let \(T : X \rightarrow Y\) be a linear continuous map. Then \(T^* : Y^* \rightarrow X^*\) is given by
\[
T^*(y^*) = y^* \circ T .
\]

The Hahn-Banach theorem shows that
\[
\|T\| = \|T^*\| .
\]
3. Subspaces, quotients and sums

**Proposition 3.1.** Let \( Y \subset X \) be a closed subset of a Banach space \( Y \) with norm \( \| \| \). Then \( (X, \| \|) \) and \( Y/X \) equipped with the norm

\[
\|y + X\| = \inf_{x \in X} \|y + x\|
\]

are Banach spaces.

**Proof.** Since \( X \) is closed, \( X \) is also complete with respect to the inherited distance. The completeness and triangle inequality for the quotient are easy (we may use the series criterion). The only important point is that \( \|y + X\| = 0 \) means that there are \( x_n \in X \) such that

\[
\lim_{n} \|y - x_n\| = 0 .
\]

This means \( y = \lim_n x_n \) is in \( X \). \( \blacksquare \)

**Definition 3.2.** Let \( Y \subset X \) be a subspace. Then

\[
Y^\perp = \{ x^* | x^* | Y = 0 \}
\]

is called the annihilator (better name). Similarly, for \( V \subset Y^* \) we may define

\[
V_\perp = \{ x \in X | \forall \phi \in V \phi(x) = 0 \} .
\]

**Proposition 3.3.** Let \( Y \subset X \) be a closed Banach subspace. Then

\[
(X/Y)^* = Y^\perp .
\]

Similarly,

\[
Y^* = X^*/Y^\perp .
\]

**Proof.** The first part is easy. Then map \( q : X \to X/Y \) is surjective, and hence \( q^* : (X/Y)^* \to X^* \) is injective. Clearly \( q^*(X/Y)^* = Y^\perp \), as sets. It remains to show that \( q^* \) is an isometry. Indeed, we see that the open unit ball in \( X/Y \) is exactly the image of \( q(\circ B_X) \) the open unit ball in \( X \) and hence

\[
\|\phi\|_{(X/Y)^*} = \sup_{\|x\| < 1} |\phi(q(x))| = \|q^*(\phi)\|_{X^*} .
\]
For the second assertion we first note that we have a map \( \sigma : X^* \to Y^* \) given by \( \sigma(\phi) = \phi|_Y \). This map is linear and continuous and \( \ker(\sigma) = Y^\perp \). By standard linear algebra we find a map
\[
\hat{\sigma} : X^*/Y^\perp \to Y^*
\]
such that \( \sigma = \hat{\sigma}q \) which is injective and contractive. We want to show that this map is bijective and isometric. For \( \phi \in Y^* \) of norm \( \leq 1 \). The Hahn-Banach theorem gives us an extension \( \psi : X \to K \) such that \( \|\psi\| = \|\phi\| \). This means \( \sigma \) is surjective. Now let \( \psi \in X^* \) such that \( \psi \in Y^\perp \) and \( \|\psi\|_{X^*/Y^\perp} = 1 \). Let \( \phi = \sigma(\psi) \) and \( \tilde{\phi} \) a Hahn-Banach extension of norm \( \alpha = \|\sigma(\psi)\| \leq 1 \). We conclude that
\[
\sigma(\tilde{\phi}) = \sigma(\phi)
\]
and hence for the quotient map \( q : X^* \to X^*/Y^\perp \)
\[
q(\phi) = q(\tilde{\phi}).
\]
Hence
\[
1 = \|\phi + Y^\perp\| = \|\tilde{\phi} + Y^\perp\| = \|\sigma(\phi)\|_{Y^*}.
\]
This means \( \hat{\sigma} \) is also isometric and hence bijective. \( \blacksquare \)

For \( 1 \leq p \leq \infty \) and two Banach spaces \( X, Y \) we may define the Banach space
\[
X \oplus_p Y
\]
with vector space \( V = X \times Y \) and norm
\[
\|(x,y)\|_p = (\|x\|^p + \|y\|^p)^{1/p}.
\]
The most common example is given by \( p = 1 \) and \( X = L_q(\Omega, \Sigma, \mu), Y = L_s(\Omega, \Sigma, \mu) \) for \( \sigma \) finite measure space.

**Definition 3.4.** Let \( Z \) be a vector space and \( T : X \to Z, S : Y \to Z \) linear maps. The subspace
\[
X + Y \subset Z
\]
is defined as
\[
X \oplus_1 Y/V
\]
where \( V = \{(x,y)|T(x) + S(y) = 0\} \) in other words \( L = T(X) + S(Y) \subset Z \) becomes a Banach space with respect to the norm
\[
\|z\| = \inf_{z = T(x) + S(y)} \{\|x\| + \|y\|\}.
\]
Corollary 3.5. The dual space of $V$ is the subspace $W$ of $X^* \oplus Y^*$ given by

$$W = \{ (x^*, y^*) | T(x) + S(y) = 0 \Rightarrow x^*(x) = y^*(y) \}.$$ 

Proof. We are just rewriting $V^\perp$. \hfill \blacksquare

HW: $K$ functional.

4. Examples

4.1. $L_p$ spaces. In the following $(\Omega, \Sigma, \mu)$ is a sigma-finite measure space. We define

$$L_0 = \{ f : \Omega \to \mathbb{R} : \lim_{\alpha \to \infty} \mu(|f| > \alpha) = 0 \}.$$ 

On $L_0$ we define the equivalence relation

$$f \sim g \text{ if } f = g \text{ } \mu \text{ a.e.}$$

i.e. there exists a set $F \in \Sigma$ with measure 0 such that $f(\omega) = g(\omega)$ for all $\omega \in F^c$. We define

$$L_0(\mu) = L_0/\sim$$

Example 4.1. For $1 \leq p < \infty$ $L_p(\omega, \Sigma, \mu) \subset L_0$ with norm

$$\| [f] \|_p = (\int |f|^p d\mu)^{1/p}$$

is a Banach space (lots of work). For $1 \leq p, \infty$ the dual is $L_p'(\Omega, \Sigma, \mu)$ (even more work).

4.2. Bounded, and compact operators. For Banach spaces $X, Y$ we have used the notation $L(X, Y)$ for the space of bounded linear operators. However, for a Hilbert space $H$ the notation $B(H) = L(H, H)$ is commonly used.

Definition 4.2. A bounded linear map $T : X \to Y$ between Banach spaces is called compact if $T(B_X)$ is totally bounded in $Y$ (or equivalently $\overline{T(B_X)}$ is compact). We denote by $K(X, Y) \subset L(X, Y)$ the subspace of compact operators.

Proposition 4.3. $K(X, Y)$ is a closed subspace of $L(X, Y)$.

Proof. The map $+: Y \times Y \to Y$, $(y, z) = y + z$ is continuous and hence for two compact sets $A, B \subset Y$ we deduce $A + B$ is again compact. This shows that
for compact $R, S : X \to Y$ the sum $R + S$ is also compact. Now let $T_n$ converge $T \in L(X, Y)$. Let $\varepsilon > 0$ and choose $\|T - T_n\|_{op} < \frac{\varepsilon}{2}$. Let $y_1, \ldots, y_m$ such that

$$T_n(B_X) \subset \bigcup_j y_j + \frac{\varepsilon}{2}B_Y.$$ 

Then

$$T(B_X) \subset \bigcup_j y_j + \varepsilon B_Y$$

and hence $T(B_X)$ is also totally bounded.

\[\square\]

**Remark 4.4.** Later: $T$ is compact if $T^*$ is compact.