Homework 6

**Due:** Friday, April 10

1. Let $1 \leq p \leq \infty$ and $u : \ell_p^n \to \ell_2^n$ be a linear isomorphism. Show that

$$\|u\|\|u^{-1}\| \;\geq\; n^{|\frac{1}{2}-\frac{1}{p}|}$$

and this lower bound can be obtained.

2. Let $T : X \to Y$ be compact and $R : X_1 \to X$, $S : Y \to Y_1$ be bounded. Show that $STR$ is compact.

3. Let $I(f)(t) = \int_0^t f(s)ds$. Show that $I : L_p([0,1]) \to L_p([0,1])$ is compact. (Hint: Look up Arzela-Ascoli and consider $I : L_1 \to C([0,1])$.)

4. Let $M_n = B(\ell_2^n)$ and $G$ be a compact group of unitaries. Let $\mu$ be the normalized Haar measure. Consider

$$\Phi(T) \;=\; \int g^* T g d\mu(g) \,.$$

Show that $g\Phi(T) = \Phi(T)g$. Equip $M_n$ with inner product $(x, y) = tr(x^* y)$. Show that $\Phi$ is the orthogonal projection onto $G' = \{T \in M_n | \forall_g Tg = gT\}$.

5. Let $H_0 \subset H$ be a subspace of a Hilbert space and $T : H \to H$ be contraction such that $T(H) \subset H_0$ and $T|_{H_0} = id$. Show that $T$ coincides with the orthogonal projection.

6. Let $u : H \to H$ be a linear isometry. Show that $u$ preserves angles

$$\text{angle}(x, y) \;=\; \cos^{-1} \frac{(x, y)}{\|x\|\|y\|} \,.$$

Are there angle preserving maps which are not isometries?

7. Let $H_0 = \{(\alpha, \alpha, \beta) | \alpha, \beta \in \mathbb{R}\}$ subset $\ell_p^3$. Find the best approximation of $(1, -1, 1)$ in $H_0$.