Homework 5

Due: Friday, March 20

1. Let X, Y be metric space. A function $f : X \to Y$ is called Baire 1- function if $f^{-1}(O)$ is a F_σ set, i.e. a countable union of closed sets. Now assume that Y is separable and X complete. Show that the sets of continuity points of a Baire-1 function are dense.

2. Let X be a Banach space such that B_{X^*} is norm separable. Let $id : (B_{X^*}, \sigma(X^*, X) \to (B_{X^*}, \| \|)$ be the identity map. Show that the map has points of continuity.

3. Let $K = B_{\ell_2}$ equipped with the weak topology. Show that $id : K \to \ell_2$ (now equipped with the norm topology) is not continuous. More precisely, show that there exists a sequence f_n such that $\|f_n\|_2 = 1$ and f_n converges to 0 weakly. Conclude that id is not continuous.

4. Let U be convex set in a locally convex topological vector space and $x \in U$ be an interior point. Let y be in the closure of U. Show that $[x, y) = \{(1-\lambda)x + \lambda y | 0 \leq \lambda < 1\}$ belongs to the interior of U.

5. On page 102 of Conway, he states that $\overline{co}(A) = \overline{co(A)}$. What does that mean?

6. We say that a Banach space Z is isomorphic to a (closed) subspace of X if there exists an injective map $u : Z \to X$ and constants $c, C > 0$ such that
$$c\|z\| \;\leq\; \|u(z)\| \;\leq\; C\|z\|$$
for all $z \in Z$. Let Y be a reflexive Banach space and X be an arbitrary Banach space. Show that X is isomorphic to a subspace of Y iff there exists a linear surjective map $T : X^* \to Y^*$.

7. We work with $\mathbb{K} = \mathbb{R}$. Find the extreme points of

   i) The unit ball in ℓ_2^n;

   ii) The unit ball in ℓ_1^n;

   iii) The unit ball of ℓ_∞^n;

iv) The unit ball of $L(\ell_1^n, \ell_1^m)$;

v) The unit ball of $L(\ell_\infty^m, \ell_\infty^n)$.

8. Show that the unit ball of $L_1([0,1])$ and c_0 has no extreme points. Conclude that none of these space is a dual space.