Homework 4

Due: Friday, March 6

i) Let $V \subset \mathbb{R}^n$ a closed, convex set which contains 0 as an interior point. Define the polar

$$V^\circ = \{y \in \mathbb{R}^n | \forall_{x \in V} (y, x) \leq 1\}$$

where $(y, x) = \sum_{j=1}^n y_j x_j$ is the usual inner product. Show that $(V^\circ)^\circ = V$.

ii) For a locally convex topological space X and an open neighborhood V of 0 show that $V^\circ = \{x^* \in X^* | \forall_{x \in V} x^*(x) \leq 1\}$ is weak*-compact (i.e. $\sigma(X^*, X)$-compact). (Conway page 131)

iii) Show that the dual of a reflexive Banach space is reflexive. Show that the subspace and the quotient of a reflexive Banach space is reflexive.

iv) Let X be a reflexive Banach space and $Y \subset X$ be a closed subspace and $x_0 \notin Y$. Show that there exists an $y_0 \in Y$ such that

$$\inf_{y \in Y} \|x_0 - y\| = \|x_0 - y_0\| .$$

A subspace $Y \subset X$ with his property is called proximal. Are there non-reflexive space which contained closed non-proximal subspaces?

v) Let X be a separable Banach space. Show that the unit ball B_{X^*} metrizable with respect to the $\sigma(X^*, X)$-topology. (Here metrizable means there is metric d which produces the same topology). Describe the metric for $X = c_0 = \{(x_n) | \lim_n x_n = 0\}$ with respect to the norm in ℓ_∞ and for $X = \ell_1$.

vi) Let (x_n) be a sequence in a metric space such that for every free ultrafilter $\mathcal{U}$ the limit

$$\lim_{n,\mathcal{U}} x_n$$

exists. Does the sequence converge? If this is not the case can you add an assumption to X which makes this true?

vii) Show that the spaces $\ell_p(\ell_q)$ are uniformly convex for $1 < p, q < \infty$.