Homework 3

Due: Friday, February 20

i) Let X be a complete topological vector space. Show that the intersection of open dense sets of X is still dense.

ii) A topological space X is called *Baire* if the intersection of open dense sets of X is still dense. A set A is called *balanced* if $x \in A$ and $|z| = 1$ implies $zx \in A$. Show that in Baire topological vector space every balanced, absorbing, closed set is a neighborhood of some point.

iii) Let K be a set and $Y = \mathbb{R}^K$ the vector space of all functions on K. We say that a subset $C \subset Y$ is *subconvex* if for every every $f_1, f_2 \in C$ and $0 \leq \lambda \leq 1$ there exists a $f \in C$ such that

$$f(x) \ \leq \ \lambda\lambda f_1(x) + (1 - \lambda) f_2(x) \ .$$

We assume in addition that for every $f \in C$

$$\sup_{x \in K} f(x) \ \geq \ 0 \ .$$

Find a functional $\phi : Y \to \mathbb{R}$ such that

a) $\phi(1_K) = 1$;

b) $f \leq 0$ implies $\phi(f) \ \geq \ 0$;

c) $\phi(f) \ \geq \ 0$ for all $f \in C$

(Hint: Consider $C^- = \{f : \sup_{x \in K} f < 0\}$ and $D^+ = \{g : \exists t > 0, g \in C \forall_{x \in K} g(x) \ \geq \ t f(x)\}$. Apply Hahn-Banach)

iv) Let ϕ be a linear functional on the real Banach space $\ell_\infty(K)$ such that

i) $\phi = \phi^+ - \phi^-$;

ii) $f \ \geq \ 0$ implies $\phi^+(f) \ \geq \ 0$ and $\phi^-(f) \ \geq \ 0$;

iii) $\|\phi\| = \|\phi^+\| + \|\phi^-\|$;

iv) $\phi(1) = 1 = \|\phi\|$.

Show that $\phi^- = 0$. Use Golstine's theorem to show that for every continuous positive functional $\phi : \ell_\infty(K) \to \mathbb{R}$ with $\phi(1) = 1 = \|\phi\|$ there is a family $(\lambda^i)_{x\in K}$ of finite supported functions such that $\lambda^i(x) \geq 0$, $\sum_x \lambda^i(x) \leq 1$ and

$$\phi(f) = \lim_i \sum_{x\in K} \lambda^i_x f(x) .$$

(You may have to choose an ultrafilter to define ϕ^+ and ϕ^-. If you can solve the problem ignoring this convergence problem, that will still give full credit).

v) Let $K \subset Z$ be a convex topological vector space and $C \subset B(K, \mathbb{R})$ a subconvex family of bounded functions such that

a) For every $f \in C$, $\{x \in K : f(x) \geq 0\}$ is closed;

b) For every $f \in C$ f is concave;

c) $\sup_x f(x) \geq 0$ for all $f \in C$.

Show that there exists an $x \in K$ such that

$$f(x) \geq 0$$

holds for all $f \in C$.

vi) Find the minmax theorem on the web (see webpage) and try to prove it for bounded functions. Is that restriction necessary?