Homework 2

Due: Friday, February 6

i) Let X be a vector space. A function $f : X \to \mathbb{R}$ is called *affine* if there exists a linear function $F : X \to \mathbb{R}$ and $r \in \mathbb{R}$ such that $f(x) = F(x) + r$. Show that a convex function $p : X \to \mathbb{R}$ satisfies

$$p(x) = \max_{f \leq p, f \text{affine}} f(x) .$$

(Hint: What can you say about $\text{ugr}(p) = \{(x,t)|p(x) < t\}$.) Can you find a convex function $p : \mathbb{R} \to \mathbb{R}$ where the maximum is obtained for two different f_1, f_2 at some point?

ii) In a topological space (X, τ) and $S \subset X$ we define the *closure* by

$$\bar{S} = \bigcap_{S \subset T, T \text{closed}} T$$

(the smallest closed set containing S).

a) Show that $\bar{S} = \{\lim_i x_i | (x_i) \subset S \text{ convergent net}\}$.

b) In a topological vector space show that

$$\bar{S} = \bigcup_{V, 0 \in V \text{ open}} S + V .$$

iii) Let X be a topological vector space and $0 \in O$ open.

a) Let $x \in X$ show that for some $\lambda > 0$ we have $x \in \lambda O$. (Such an O is called absorbing.)

b) Show that X admits a neighborhood basis of 0 consisting of closed sets (Hint: $W \subset \bar{W} \subset 2W$).

iv) Let X, Y be topological vector spaces and $L(X, Y)$ be the set of continuous linear maps. Show that the family

$$B_{V,W} = \{T : X \to Y : T(V) \subset W\}$$

index by open neighborhoods $0 \in V \subset X$, $0 \in W \subset Y$, defines a neighborhood basis at 0. Show that $L(X, Y)$ equipped with the corresponding topology is again a topological vector space.

v) Let Y be a topological vector space. A net (y_i) is called *Cauchy* if for every $0 \in V$ open there exists an i_0 such that

$$y_i - y_j \in V$$

for all $i, j > i_0$. A topological vector space Y is called *complete* if every Cauchy net is convergent. Show that for a complete topological vector space Y and any topological vector space X the space $L(X, Y)$, equipped with the topology from above, is again complete.

(Hint: a) Let (T_i) a Cauchy net and show that $T(x) = \lim_i T_i(x)$ is a linear map. b) Assume that $(T_i - T_j)(O) \subset W$. Show that

$$T(x) - T_i(x) \in \bar{W} \; .$$

c) Conclude that T is continuous and convergence takes place.