Homework 541-1

Due date: January 30

i) Let I be an infinite set. In this exercise you may use standard facts on cardinality: $|I \times I| = |I| =$ the cardinality of finite subsets of I, and whatever you find.

   i) Let $V = \mathbb{K}^I$, $W = \mathbb{K}^J$ and $A : V \to W$ be a linear map viewed as an infinite matrix a_{ij}. Show that
   $$J' = \{j \in J | \text{for some } i \in I\}$$
   satisfies $|J'| \leq |I|$

   ii) Show that for a vector space over $\mathbb{K}$ the cardinality of the basis is independent of the choice of the basis.

ii) Let (X, τ) be a topological vector space. On the product space X^I we define
$$\prod O_i \; = \; \{f : I \to X \; : \; f(i) \in O_i\} \, .$$
If O_i is open for all $i \in I$ and $O_i = X$ for all but finitely many i, we declare $\prod O_i$ to be open. Show that this defines a topology τ such that all the projections maps
$$p_i : X^I \to X \, , \; p_i(f) = f(i)$$
are continuous. Conversely given that you have another topology τ' such that all the projection maps are continuous. What can you say?

iii) Let $Q_{k,j} = [2k, 2k+1] \times [2j, 2j+1]$. a) Show that there exists a topology (=collection of open sets) which contains alls usual open sets and the square $Q_{k,j}$. b) Show that this topology can not come from a topological vector space.

iv) Show that in for a topological vector space (V, τ) and $x \neq y$ you can find open sets O_x and O_y such that $x \in O_x$, $y \in O_y$ and
$$O_x \cap O_y \; = \; \emptyset \, .$$
Hint: show that for every open neighborhood W, you can find an open neighborhood V such that $V - V \subset W$.

v) Let (X, d) be a metric space. Show that an open ball is open and a closed ball is closed. Show that a metric space is always Haussdorff (goolge it).