Mathematical Programming 34 (1986) 232-234
North-Holland

ON FAN'S MINIMAX THEOREM

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Received 28 September 1984
Revised manuscript received 29 March 1985

A new brief proof of Fan's minimax theorem for convex-concave like functions is established using separation arguments.

Key words: Convex-Concave Like Functions, Fan's Minimax Theorem, Separation Theorem.

Recall that $f: X \times Y \to \mathbb{R}$ is said to be *convex-concave like* on $X \times Y$ if, for $0 \leq t \leq 1$,

(a) for x_1, x_2 in X there exists x_3 in X with

$$f(x_3, y) \leq tf(x_1, y) + (1-t)f(x_2, y) \tag{1}$$

for all y in Y; and

(b) for y_1, y_2 in Y there exists y_3 in Y with

$$f(x, y_3) \geq tf(x, y_1) + (1-t)f(x, y_2) \tag{2}$$

for all x in X, [4].

The general form of Fan's minimax theorem [2, 4] is:

Theorem A. *Suppose that X and Y are non-empty sets with f convex-concave like on $X \times Y$. Suppose that X is compact and $f(\cdot, y)$ is lower semicontinuous on X for each y in Y. Then*

$$p := \min_X \sup_Y f(x, y) = \sup_Y \min_X f(x, y). \tag{3}$$

Here, it is possible that $p = \pm\infty$, in which case 'min' must be appropriately read. If Y is actually compact and $f(x, \cdot)$ is upper semicontinuous on Y for each x in X, then 'sup' may be replaced by 'max' in (3). There is of course a dual form.

The purpose of this note is to produce a very brief, self-contained proof of Theorem A.

Proof. If $p = -\infty$ there is nothing to prove since

$$\inf_X \sup_Y f(x, y) \geq \sup_Y \inf_X f(x, y)$$

always holds.

Let α be any real number strictly less than p. Since $C(y) := \{x \in X \mid f(x, y) \leqslant \alpha\}$ is compact for each y, and since $\bigcap\{C(y) \mid y \in Y\} = \emptyset$, one can find $y_1, y_2, \ldots, y_n$ in Y such that $\alpha < \min_X \sup_{1 \leqslant i \leqslant n} f(x, y_i)$.

Now consider

$$E := \{(z, r) \in \mathbb{R}^{n+1} \mid \exists x \in X, f(x, y_i) \leqslant r + z_i, i = 1, 2, \ldots, n\}. \tag{4}$$

Since $f(\cdot, y)$ is convex like, it follows easily that E is convex. Clearly, $(0, 1 + \max_{1 \leqslant i \leqslant n} f(x, y_i))$ is interior to E for any x in X. Also by construction

$$(0, \alpha) \notin E. \tag{5}$$

By the Separation Theorem [3] one can find $(\lambda_1, \lambda_2, \ldots, \lambda_n, \bar{r}) \neq 0$ with

$$\sum_{i=1}^{n} \lambda_i z_i + \bar{r} r \geqslant \bar{r} \alpha \quad \text{for } (z, r) \in E. \tag{6}$$

One observes that $\lambda_i \geqslant 0$, $\bar{r} \geqslant 0$ because $E + \mathbb{R}_+^{n+1} \subset E$. Also $\bar{r} > 0$ as $(0, 1 + \max_{1 \leqslant i \leqslant n} f(x, y_i))$ lies in int E. Thus we have

$$\sum_{i=1}^{n} \frac{\lambda_i}{\bar{r}} f(x, y_i) + \left(\sum_{i=1}^{n} \frac{\lambda_i}{\bar{r}} - 1 \right) r \geqslant \alpha \tag{7}$$

for all x in X and r in $\mathbb{R}$ because $(f(x, y_i) + r, -r) \in E$. Hence

$$\sum_{i=1}^{n} \frac{\lambda_i}{\bar{r}} = 1$$

and, as $f(x, \cdot)$ is concave like, (7) shows that, for some $\hat{y}$ in Y,

$$f(x, \hat{y}) \geqslant \alpha \quad \text{for all } x \text{ in } X.$$

Thus

$$\sup_Y \inf_X f(x, y) \geqslant \alpha$$

and, since α is arbitrary less than p,

$$\sup_Y \inf_X f(x, y) = \inf_X \sup_Y f(x, y). \tag{8}$$

Now (3) follows as $f(x, y)$ and $\sup_Y f(x, y)$ are lower semi-continuous on X which is compact. □

Equivalently, one can consider

$$h(z) := \inf_X \{r \mid f(x, y_i) \leqslant z_i + r, i = 1, 2, \ldots, n\} \tag{9}$$

and observe that, while f is not convex, h is. Since h is continuous at 0, it possesses a subgradient [3]; which leads again to (7). Developments related to this can be found in [1].

Acknowledgements

The authors would like to thank the referee for a number of suggestions that improved the final version of the note.

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