1. The space of integrable functions

In the following \((\Omega, \Sigma, \mu)\) is a measure space.

Definition 1.1. \(f \in S(\mu)\) if \(f : \Omega \to \mathbb{R}\) is a measurable function such that \(f(\Omega)\) is a finite set and

\[ \mu(f \neq 0) < \infty. \]

Then

\[ I(f) = \sum_{0 \neq r \in f(\Omega)} r \mu(\{f = r\}). \]

Proposition 1.2. \(I : S(\mu) \to \mathbb{R}\) is linear. Moreover, \(f \leq g\) implies \(I(f) \leq I(g)\).

Furthermore, we have the triangle inequality

\[ I(|f - g|) \leq I(|f - h|) + I(|h - g|). \]

Proof. 1. Note that \(I(\lambda g) = \lambda I(g)\). Now if \(f, g \in S(\mu)\), then

\[ f = \sum_{i=0}^{n} x_i 1_{E_i} \text{ where } E_i = f^{-1}(\{x_i\}) \]

\[ g = \sum_{i=0}^{n} y_i 1_{F_i} \text{ where } F_i = f^{-1}(\{y_i\}), \]

where \(x_0 = y_0 = 0\). We may assume that

\[ i \neq k \Rightarrow E_i \cap E_k = \emptyset \cup E_i = \Omega. \]

Moreover, for \(i \neq 0\) we know that \(\mu(E_i) < \infty\). Similarly, we may assume

\[ j \neq l \Rightarrow F_j \cap F_l = \emptyset \cup F_j = \Omega. \]

Consider

\[ \{x_i + y_j : 0 \leq i \leq n, 0 \leq j \leq m\} = \{Z_0, Z_1, \ldots, Z_N\}. \]

We assume that \(Z_0 = 0\). Note that \(E_0 \cap F_0 \subset (f + g)^{-1}(0)\). Then

\[ I(f + g) = \sum_{r=1}^{N} Z_r \mu(\cup_{x_i + y_j = Z_r} E_i \cap F_j) \]

\[ = \sum_{i,j, \min(i,j) > 0} (x_i + y_j) \mu(E_i \cap F_j) \]

\[ = \sum_{i=1}^{n} x_i \sum_{j=0}^{m} \mu(E_i \cap F_j) + \sum_{j=1}^{m} y_j \sum_{i=0}^{n} \mu(E_i \cap F_j) \]

\[ = \sum_{i} x_i \mu(E_i) + \sum_{j} y_j \mu(F_j) = I(f) + I(g). \]
2. If \( f \leq g \), then \( E_i \cap F_j \neq \emptyset \) implies \( x_i \leq y_j \). Let \( G = \bigcup_{i=1}^{n} E_i \). This yields

\[
I(f) = \sum_{i=1}^{n} x_i \mu(E_i) = \sum_{i=1}^{n} x_i \sum_{j=0, E_i \cap F_j \neq \emptyset}^{m} \mu(E_i \cap F_j)
\]

\[
\leq \sum_{j=1}^{m} y_j \mu(E_i \cap F_j) = \sum_{j=0}^{m} y_j \sum_{i=1}^{m} \mu(E_i \cap F_j)
\]

\[
= \sum_{j=1}^{m} y_j \sum_{i=1}^{n} \mu(E_i \cap F_j) = \sum_{j=1}^{m} y_j \mu(F_j \cap G)
\]

\[
= I(g1_G).
\]

However, if \( \omega \in G^c \), then \( f(\omega) = 0 \) and hence \( g(\omega) \geq 0 \). This yields

\[
I(g1_G^c) \geq 0.
\]

Therefore \( I(f) \leq I(g1_G) \leq I(g1_G) + I(g1_G^c) = I(g) \).

3. We note \( f - g \leq |f - g| \) and hence

\[
I(f) - I(g) = |I(f - g)| \leq I(|f - g|).
\]

Similarly, \( I(g) - I(f) \leq I(|g - f|) = I(|f - g|) \). The assertion follows.

**Definition 1.3.** Let \( f : \Omega \rightarrow [0, \infty] \) a positive measurable function. Then

\[
I(f) = \sup_{0 \leq h \leq f, h \in S(\mu)} I(h).
\]

\( f \) is called (positive) integrable if \( I(f) \) is finite.

**Lemma 1.4.** 1) \( f \leq g \), then \( I(f) \leq I(g) \). 2) \( I(f + g) \geq I(f) + I(g) \)

**Proof.** 2) Let \( 0 \leq f_n \leq f \) and \( 0 \leq g_n \leq g \Rightarrow 0 \leq f_n + g_n \leq f + g \). Then,

\[
I(f) + I(g) = \sup I(f_n) + \sup I(g_n) \leq I(f + g).
\]

We want to show that for integrable \( f, g \) we have \( I(f + g) \leq I(f) + I(g) \). This will be done in several steps.

**Lemma 1.5.** \( f \geq 0 \) integrable, \( 0 \leq h \leq f, h \in S(\mu) \). Then

\[
I(f) = I(f - h) + I(h).
\]

**Proof.** Let \( \epsilon > 0 \) and \( 0 \leq \tilde{g} \leq f, \tilde{g} \in S(\mu) \) such that

\[
I(\tilde{g}) \leq I(f) \leq I(\tilde{g}) + \epsilon.
\]
1. THE SPACE OF INTEGRABLE FUNCTIONS

Define \( g = \max\{\tilde{g}, h\} \geq \tilde{g} \), then

\[
I(g) \leq I(f) \leq I(\tilde{g}) + \epsilon \leq I(g - h) + I(h) + \epsilon \leq I(f - h) + I(h) + \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, this concludes the proof.

**Lemma 1.6.** \( f \geq 0 \) integrable. Then there exits a sequence \( f_n \in S(\mu) \) such that \( 0 \leq f_n \leq f \), \( f_n \) is increasing and

\[
I(f - f_n) \leq 4^{-n}.
\]

**Proof.** Let \( \epsilon = 2^{-n} \). Furthermore, let \( 0 \leq \tilde{g}_n \leq f \) and \( I(f) \leq I(\tilde{g}_n) + \epsilon_n \). Define \( g_n = \max\{\tilde{g}_1, \ldots, \tilde{g}_n\} \), then

\[
I(g_n) \leq I(f) \leq I(\tilde{g}_n) + \epsilon_n \leq I(g_n) + \epsilon_n.
\]

By Lemma 1.5:

\[
I(g_n) + \epsilon_n \geq I(f) = I(f - g_n) + I(g_n).
\]

Subtracting \( I(g_n) \) yields \( I(f - g_n) < \epsilon_n \).

**Lemma 1.7.** (Chebychev) \( f \) integrable, \( \lambda > 0 \). Then

\[
\lambda \mu(f \geq \lambda) \leq I(f).
\]

**Proof.** Let \( h = \lambda 1_{f \geq \lambda} \leq f \) and let \( \Omega_n \subset \Omega \) an increasing sequences of subsets such that \( \cup \Omega_n = \Omega \). Then

\[
h_n = \lambda 1_{f_n \geq \lambda} \leq I(f),
\]

so

\[
\mu(f \geq \lambda) = \lim_n \mu(\{f \geq \lambda\} \cap \Omega_n) = \lim_n \frac{1}{\lambda} I(h_n) \leq \frac{1}{\lambda} I(f).
\]

**Lemma 1.8.** Let \( (f_n) \) and \( f \) be measurable such that

\[
I(|f_n - f|) \leq 4^{-n}
\]

Then \( f_n \) converges to \( f \) almost everywhere.

**Proof.** Let \( E_n = \{\omega : |f - f_n| > 2^{-n}\} \), \( F_n = \cup_{k \geq n} E_k \) and \( F = \cap_n F_n \), then

\[
\mu(F) \leq \lim_n \mu(F_n) \leq \lim_n \sum_{k \geq n} \mu(E_k) = \lim \sum_{k \geq n} 2^{-k} = \lim 2^{-n} = 0.
\]

If \( \omega \notin F \Rightarrow \exists n \) such that \( \forall k \geq n, f(\omega) - 2^{-k} \leq f_k(\omega) \leq f(\omega) + 2^{-k} \), which implies

\[
\lim f_k(\omega) = f(\omega).
\]
Lemma 1.9. (Beppo-Levi) \(0 \leq f_n \leq f_{n+1}\) all integrable such that \(\lim_n I(f_n) < \infty\). If \(f_n\) converges to \(f\) almost everywhere, then 
\[
I(f) \leq \lim_n I(f_n)
\]

Proof. Let \(0 \leq h \leq f, h \in S(\mu)\) and 
\[
h = \sum_{i=1}^{m} r_i 1_{E_i}
\]
Let \(F \subset \Omega\) such that \(\lim f_n(\omega) = f(\omega) \forall \omega \in F\). Given any \(\epsilon > 0\), define 
\[
E_{i,n} = \left\{ \omega \in E_i \cap F : f_n(\omega) > \frac{r_i}{1 + \epsilon} \right\} \subset E_i,
\]
then \(E_i \cap F = \cup_n E_{i,n}\) because \(\lim_n f_n(\omega) = f(\omega) \geq r_i\). We may find \(n\) such that 
\[
\mu(E_{i,n}) \geq \frac{\mu(E_i \cap F)}{1 + \epsilon}
\]
for all \(i = 1, \ldots, m\). Then define 
\[
h^{\epsilon,n} = \sum_{i} r_i \frac{1}{1 + \epsilon} 1_{E_{i,n}} \leq f_n 1_F \leq f_n.
\]
Note that \(\leq f_n 1_F \leq f_n\) and hence 
\[
I(h^{\epsilon,n}) \leq I(f_n) \leq \lim_n I(f_n).
\]
Moreover, 
\[
I(h) = \sum_{i=1}^{m} r_i \mu(E_i) = \sum_{i=1}^{m} r_i \mu(E_i \cap F)
\]
\[
\leq (1 + \epsilon)^2 \sum_{i=1}^{m} r_i \frac{1}{1 + \epsilon} \mu(E_{i,n})
\]
\[
\leq (1 + \epsilon)^2 I(h^{\epsilon,n}) \leq (1 + \epsilon)^2 I(f_n) \leq (1 + \epsilon)^2 \lim_n I(f_n).
\]
Since \(\epsilon > 0\) is arbitrary, we get \(I(h) \leq \lim_n I(f_n)\). Taking the supremum, we deduce the assertion. \(\blacksquare\)

Corollary 1.10. (Fatou) Let \(0 \leq f_n\) be positive integrable functions. Then 
\[
I(\liminf_n f_n) \leq \liminf_n I(f_n).
\]
Proof. The sequence \(g_n = \inf_{m \geq n} f_m\) is increasing. Then 
\[
I(\sup_n g_n) = \sup_n I(g_n) \leq \sup_n \inf_m I(f_m) = \liminf_n I(f_n).
\]
\(\blacksquare\)

Proposition 1.11. \(f, g\) positive integrable. Then 
\[
I(f) + I(g) = I(f + g).
\]
1. THE SPACE OF INTEGRABLE FUNCTIONS

Proof. Let $f_n \in S(\mu)$, $g_n \in S(\mu)$ increasing sequences such that $I(f - f_n) \leq 4^{-n}$ and $I(g - g_n) \leq 4^{-n}$. Then the sequence $h_n = f_n + g_n$ converges to $f + g$ almost everywhere and

$$I(f + g) \leq \lim_n I(f_n + g_n) = \lim_n I(f_n) + I(g_n) = I(f) + I(g).$$

Definition 1.12. A measurable function $f : \Omega \to [-\infty, \infty]$ is called integrable if there exists a sequence $(f_n)$ in $S(\mu)$ such that

$$\lim_n I(|f - f_n|) = 0.$$

We denote $I(\mu)$ the space of integrable functions

Proposition 1.13. Let $f$ be $\mu$-integrable and $(f_n), (f'_n)$ such that

$$\lim_n I(|f - f_n|) = 0 = \lim_n I(|f - f'_n|)$$

Then

$$\lim_n I(|f_n - f'_n|) = 0.$$

In particular,

$$\int f = \lim_n I(f_n)$$

is well-defined.

Lemma 1.14. Let $f \geq 0$ be $\mu$-integrable. Then

$$I(f) = \int f.$$

Proof. Let $(f_n)$ be a sequence of simple functions such that $I(|f - f_n|) \leq 4^{-n}$. Then $f_n$ converges to $f$ almost everywhere. For fixed $n \in \mathbb{N}$ we consider $E_n = \{\omega : f_n(\omega) < 0$. Then

$$1_{E_n}|f - f_n| = 1_{E_n}|f| + 1_{E_n}|f_n| \geq 1_{E_n}|f|.$$

Thus we have $I(|f - f_n^+|) \leq I(|f - f_n|) \leq 4^{-n}$. Then $f_n^+$ converges to $f \mu$-almost everywhere and Fatou’s lemma implies

$$I(f) \leq \liminf_n I(f_n^+) \leq \sup_n I(|f_n|).$$

However,

$$|I(|f_n|) - I(|f_m|)| = |I(|f_n|) - I(|f_m|)| \leq I(||f_n| - |f_m||) \leq I(|f_n - f_m|) \leq I(|f_n - f|) + I(|f - f_m|) \leq 4^{-n} + 4^{-m}.$$
Thus $I(|f_n|)$ is Cauchy and hence $\sup_n I(|f_n|)$ bounded. We get

$$I(h) \leq \sup_n I(|f_n|).$$

In particular, $I(f)$ is finite. Equality follows from the preceding Proposition.

\section*{2. Convergence Theorems and applications}

\textbf{Lemma 2.1.} (Fatou) Let $(f_n)$ be positive integrable functions. Then

$$\int \liminf_n f_n \leq \liminf_n \int f_n.$$
Theorem 2.2. (Dominated convergence theorems) Let \( f \geq 0 \) be positive integrable function. Let \((g_n)\) be integrable functions such that
\[
|g_n| \leq f \quad \mu \text{ a.e.}
\]
for all \( n \in \mathbb{N} \) and \( g = \lim_n g_n \) exists. Then \( g \) is integrable and
\[
\int g = \lim_n \int g_n.
\]

**Proof.** Let us first assume \( |g_n| \leq f \) everywhere and \( g = \lim_n g_n \). Then the sequence \( h_n = g_n + f \) is positive. By Fatou’s Lemma, we find
\[
\int (g + f) = \int \lim_n g_n + f \leq \liminf_n \int g_n + \int f.
\]
Thus \( g + f \) and hence \( g \) is integrable. Subtracting \( \int f \) we get
\[
\int g \leq \liminf_n \int g_n.
\]
Now, we consider \( k_n = -g_n + f \) and deduce similarly as before
\[
-\int g + \int f \leq \liminf_n \int -g_n + \int f.
\]
Thus
\[
\limsup_n \int g_n \leq \int g.
\]
In the general case, we consider the exceptional set \( E_n \in \Sigma \) of measure 0 such that \( |g_n(\omega)| \leq f(\omega) \) for all \( \omega \in E_n^c \). Let \( F \in \Sigma \) be of measure 0 such that \( \lim_n g_n(\omega) = g(\omega) \) holds for \( \omega \in F^c \). We define \( E = F \cup \bigcup_n E_n \). Then, we have \( \mu(E) = 0 \). Moreover, the functions \( \tilde{g}_n = g_n1_{E^c} \) and \( \tilde{g} = g1_{E^c} \) satisfy all the requirements above. The assertion follows from the following remark. \( \blacksquare \)

Remark 2.3. Let \( E \) be a set of measure 0 and \( f : \Omega \to 0 \) be a measurable function, then \( \int |f|1_E = 0 \).

**Proof.** Let \( 0 \leq h \leq |f|1_E \). We may write
\[
h = \sum_{i=1}^m r_i1_{F_i}.
\]
Then \( h1_E = h \) and hence
\[
h = \sum_{i=1}^m r_i1_{E \cap F_i}.
\]
This yields \( I(h) = 0 \). \( \blacksquare \)

We will now discuss an application.
**Theorem 2.4.** (Riemann-Lebesgue lemma) Let \( f : \mathbb{R} \to [-\infty, \infty] \) be integrable. Then
\[
\lim_{k \to \infty} \int \cos(kt) f(t) \, dt = 0.
\]

**Lemma 2.5.** Let \( f \) be a step function. Then
\[
\lim_{k \to \infty} \int \cos(kt) f(t) \, dt = 0.
\]

**Proof.** Let \( f = \sum_{i=1}^{m} r_i 1_{[a_i,b_i]} \). By linearity it suffices to show that
\[
\lim_{k \to \infty} \int_{a}^{b} \cos(tk) dt = 0.
\]

This follows obviously from
\[
\left| \int_{a}^{b} \cos(tk) dt \right| = \frac{\left| \sin(bkt) - \sin(akt) \right|}{k} \leq \frac{2}{k}.
\]

The assertion is proved.

The Riemann-Lebesgue lemma is an easy consequence of the following result.

**Theorem 2.6.** Let \( f : \mathbb{R} \to [-\infty, \infty] \) be integrable and \( \varepsilon > 0 \). Then there exits a simple function \( h \) such that
\[
\int \left| f - h \right| < \varepsilon.
\]

**Proof of Theorem 2.4 from Theorem 2.6.** Let \( h \) be a simple function with \( \int \left| f - h \right| < \frac{\varepsilon}{2} \). Let \( k_0 \) such that
\[
\left| \int \cos(kt)h(t) dt \right| < \frac{\varepsilon}{2}
\]
for all \( k > k_0 \). Then
\[
\left| \int \cos(kt)f(t) dt \right| \leq \left| \int \cos(kt)(f(t) - h(t)) dt \right| + \left| \int \cos(kt)h(t) dt \right|
\]
\[
< \int \left| f - h \right| + \frac{\varepsilon}{2} < \varepsilon
\]
holds for all \( k > k_0 \).

The following Lemma is an immediate application of the dominated convergence theorem:

**Lemma 2.7.** Let \( f : \mathbb{R} \to [-\infty, \infty] \) be integrable and \( \varepsilon > 0 \). Then there exists \( n \in \mathbb{N} \) such that
\[
\int_{|x| \geq n} |f| < \varepsilon.
\]
Proof of Theorem 2.6. Let $\varepsilon > 0$. We choose $n \in \mathbb{N}$ such that

$$\int_{|x| \geq n} |f| < \frac{\varepsilon}{3}.$$ 

Let $h : [-n, n] \rightarrow \mathbb{R}$ a simple function such that

$$\int_{-n}^{n} |f - h| < \frac{\varepsilon}{3}.$$ 

Let $C = \sup |h|$ and $\delta = \frac{\varepsilon}{6(C + n)}$. We apply the consequence of Lusin’s theorem and find a simple function $g$ such that

$$m(|g - h| > \delta) < \delta.$$ 

Moreover, the construction yields such an $h$ with $|h| \leq C$. Then, we get

$$\int_{-n}^{n} |g - h| = \int_{-n}^{n} 1_{|g - h| > \delta} |g - h| + \int_{-n}^{n} 1_{|g - h| \leq \delta} |g - h|$$

$$\leq 2Cm(|g - h| > \delta) + 2n\delta \leq 2(C + n)\delta < \frac{\varepsilon}{3}.$$ 

We insist that $h = h_{1_{[-n, n]}}$. Thus we get

$$\int |f - h| \leq \int_{|x| > n} |f| + \int_{-n}^{n} |f - h| \leq \int_{|x| > n} |f| + \int_{-n}^{n} |f - g| + \int_{-n}^{n} |g - h| < \varepsilon.$$ 

Lemma 2.8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a step function and $\varepsilon > 0$. Then there exists a continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $\lim_{|x| \to \infty} |g(x)| = 0$ and

$$\int |f - g| < \varepsilon.$$ 

Proof. Let $f = 1_{[a, b]}$ and $0 < \delta < b - a$. Then

$$g_{\delta, a, b}(t) = \begin{cases} 
\delta^{-1}(t - (a - \delta)) & \text{if } a - \delta \leq t \leq a \\
1 & \text{if } a \leq t \leq b \\
1 - \delta^{-1}(t - b) & \text{if } b \leq t \leq b + \delta \\
0 & \text{else}
\end{cases}.$$ 

Then $g$ is continuous and

$$\int |g_{\delta, a, b} - 1_{[a, b]}|d\mu \leq 2\int_{0}^{\delta} tdt = \delta.$$
For an arbitrary simple function \( f = \sum_{i=1}^{n} r_i 1_{[a_i, b_i]} \) we consider \( g_\delta = \sum_{i=1}^{n} r_i g_{\delta, a_i, b_i} \).

Then we get

\[
\int |f - g_\delta| \leq \sum_{i=1}^{n} |r_i| \int |1_{[a_i, b_i]} - g_{\delta, a_i, b_i}(t)| \leq \sum_{i=1}^{n} |r_i| \delta.
\]

Thus \( \delta < \frac{\varepsilon}{1 + \sum_{i=1}^{n} |r_i|} \) implies the assertion.

\[\square\]

**Corollary 2.9.** Let \( f : \mathbb{R} \to [-\infty, \infty] \) be an integrable function and \( \varepsilon > 0 \). Then there exists a continuous function \( g \) vanishing at \( \pm \infty \) such that

\[
\int |f - g| d\mu < \varepsilon.
\]

**Proof.** Let \( h \) be a step function such that

\[
\int |f - h| d\mu < \frac{\varepsilon}{2}.
\]

Let \( g \) be a continuous function (constructed above) such that \( \int |g - h| < \frac{\varepsilon}{2} \). This function \( g \) vanishes for large \( x \)'s and satisfies

\[
\int |f - g| d\mu \leq \int |f - h| d\mu + \int |h - g| < \varepsilon.
\]

This proves the assertion.

\[\square\]