19. The Lebesgue Differentiation Theorem

**Theorem 19.1 (Lebesgue Differentiation).** Let $f$ be an integrable function on $\mathbb{R}$. Then each of the following equalities holds almost everywhere on $\mathbb{R}$.

\[
\lim_{r \to 0^+} \frac{1}{2r} \int_{[x-r,x+r]} f \, dm = f(x),
\]
\[
\lim_{r \to 0^+} \frac{1}{r} \int_{[x,x+r]} f \, dm = f(x),
\]
\[
\lim_{r \to 0^+} \frac{1}{r} \int_{[x-r,x]} f \, dm = f(x).
\]

**Proof.** We need only prove the result for $f \geq 0$ and $x$ in a fixed, bounded, open interval $J$. By Lusin’s Theorem, for any $\varepsilon > 0$, there is a compact set $K \subset J$ with $m(J \setminus K) < \varepsilon$ such that $f|_K$ is continuous on $K$. Let $O = \mathbb{R} \setminus K$ be the complement of $K$ and $a = \inf_K$, $b = \sup_K$. We may write $O = \bigcup_j I_j$ as a countable union of open intervals. We may assume $I_1 = (-\infty, a)$. There we define $h(x) = f(a) \frac{|a|^2}{|x|^2}$. On $I_2 = (b, \infty)$ we define $h(x) = f(b) \frac{|b|^2}{|x|^2}$. All the other components are bounded. Now we replace inductively every $I_j$ by the maximal interval $I'_j \supset I_j$ such that $I'_j \cap K = \emptyset$. On such an $I'_j$ we may use a piecewise linear function. In this way we construct a continuous function $h$ such that $h|_K = f|_K$. Note that $g = h - 1_K f$ is measurable and integrable by construction. Moreover, $g$ vanishes on $K$. By Proposition 20.3 all the limits for $g$ vanish on $K$. Thus we see that

\[ f = h - g + f 1_{\mathbb{R} \setminus K} \]

holds on $\mathbb{R}$. By the continuity of $h$, each of the limit results holds almost everywhere on $K$ for $h$, $g$ and $f \cdot 1_{\mathbb{R} \setminus K}$, and thus for $f$. Since $\varepsilon$ is arbitrary, the limit result is established for almost all points of $J$.

We have already established that if $f$ is Lebesgue integrable on $[a, b]$, then $F(x) := \int_a^x f \, dm = \int_{[a,x]} f \, dm$ is continuous. We used the fact that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $mA < \delta$, then $\int_A |f| < \varepsilon$. It was important, however, that points have measure 0. For differentiablility, we have the following results that generalizes part of the Fundamental Theorem of Calculus. It uses the fact that we are integrating with respect to Lebesgue measure, so that the length of an interval is its measure.

**Theorem 19.2.** Suppose $f$ is Lebesgue integrable on $[a, b]$, and $F(x) = \int_a^x f \, dm + C$ where $C$ is a constant. Then $F'(x) = f(x)$ for almost all $x \in [a, b]$. 
Proof. For $\Delta x = r > 0$ and $x + \Delta x \leq b,$

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \frac{1}{r} \int_{[x,x+r]} f \, dm.$$ 

For $\Delta x = -r < 0$ and $x + \Delta x \geq a,$

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = \frac{1}{-r} \cdot \int_{[x-r,x]} f \, dm = \frac{1}{r} \int_{[x-r,x]} f \, dm.$$ 

The result now follows from the previous theorem.

Since $\int_a^b f = 0,$ we can think of the constant $C$ as $F(a)$. The theorem says, we can differentiate the indefinite integral of a Lebesgue integrable function and get back the integrand almost everywhere. We would like to integrate the derivative of any reasonable function and get back the function we started with. We know this is impossible with the Cantor function since the Cantor function’s derivative is 0 almost everywhere.
20. A covering lemma and the local maximal function

Among the most important tools in analysis are the results known as covering theorems. For the real line, the covering sets are intervals. In this section, we present a simple form of the best covering theorem for the real line. The original result is an extension by J. Aldaz [“A general covering lemma for the real line”, Real Anal. Exchange 17 (1991/92), 394–398] of a lemma of T. Radó (“Sur un problème relatif à un théorème de Vitali”, Fundamenta Mathematicae 11 (1928), 228–229.) In the Aldaz result, the constant 3 is improved to $2 + \varepsilon$ for an arbitrary $\varepsilon > 0$, and the result is valid for any finite Borel measure, not just Lebesgue measure $m$ on a bounded interval.

**Theorem 20.1 (Rado-Aldaz).** Given an arbitrary collection $\mathcal{I}$ of non-degenerate intervals, all contained in a fixed bounded interval $J$, the set $\bigcup_{I \in \mathcal{I}} I$ is measurable, and there is a finite disjoint subset $\{I_1, \ldots, I_n\} \subseteq \mathcal{I}$ such that

$$m(\bigcup_{I \in \mathcal{I}} I) \leq 3 \cdot \sum_{k=1}^{n} m(I_k).$$

**Proof.** Let $B$ be the set of those right-hand end points of intervals $I \in \mathcal{I}$ such that $b \in I$ but $b$ is not in the interior of any interval of $\mathcal{I}$. Then for some $\delta > 0$, $(b - \delta, b) \subset I$ and $(b - \delta, b] \cap B = \{b\}$. We may associate a rational number in the interval $(b - \delta, b)$ with $b$. It follows that $B$ is a countable set. A similar fact is true for left-hand end points that are not interior to any interval in $\mathcal{I}$. Therefore, $(\bigcup_{I \in \mathcal{I}} I) \setminus (\bigcup_{I \in \mathcal{I}} I^o)$ is at most a countable set. By Lindelöf’s theorem (i.e., any union of open intervals equals the union of a countable subcollection) we may assume that $\mathcal{I}$ itself is a countable collection $\{I_n\}$, whence $\bigcup_{I \in \mathcal{I}} I = \bigcup_{n=1}^{\infty} I_n$ is measurable.

Now since all the intervals are contained in the bounded interval $J$,

$$m(\bigcup_{n=1}^{\infty} I_n) = \lim_{N \to \infty} m(\bigcup_{n=1}^{N} I_n) < +\infty.$$

We employ Rado’s result after first choosing $N$ so that

$$\frac{3}{2} \cdot m(\bigcup_{n=1}^{N} I_n) \geq m(\bigcup_{n=1}^{\infty} I_n) = m(\bigcup_{I \in \mathcal{I}} I).$$

Ordering the finite collection, we discard the first interval if it is covered by the remaining intervals. Otherwise, we keep the first interval and consider the second. In either case, this does not change the measure $m(\bigcup_{n=1}^{N} I)$. Continuing in this way, we may assume that each $I_n$ in our finite collection contains a point $x$ not in any other interval of the collection. Now, we order these points and reorder the corresponding intervals so that for any indices $i$, $j$, and $k$ with $i < j < k$ we have
$x_i < x_j < x_k$ and thus $I_i \subseteq (\infty, x_j)$ and $I_k \subseteq (x_j, +\infty)$. That is, the points are given the ordering inherited from $\mathbb{R}$, and the intervals are given the same ordering as their points. Since the intervals with even indices form a disjoint collection, as do the intervals with odd indices, the desired subset of $\mathcal{I}$ is whichever of these two families has the greater total measure. For example, if we choose the even indices, then

$$3 \cdot m(\cup I_{2n}) = \frac{3}{2} \cdot 2 \cdot m(\cup I_{2n}) \geq \frac{3}{2} \cdot m(\cup_{n=1}^N I_n) \geq m(\cup_{I \in \mathcal{I}} I).$$

The next result is refined maximal inequality due to Jürgen Bliedtner and P. Loeb ("Limit Theorems via Local Maximal Functions", preprint.) Here, we let $f$ be a nonnegative integrable function on $\mathbb{R}$. We set $\mathcal{I}(x, r)$ equal to the set of intervals $I$ containing $x$ with positive length $m(I) \leq r$, and we set

$$M(f, r, x) := \sup_{I \in \mathcal{I}(x, r)} \frac{1}{m(I)} \int_I f \, dm.$$

Since $M(f, r, x)$ decreases as $r$ decreases, we may set

$$M(f, x) := \lim_{r \to 0^+} M(f, r, x),$$

where the limit is understood to be $+\infty$ if $M(f, r, x) = +\infty$ for all $r > 0$.

**Proposition 20.2.** Let $E$ be a bounded subset of $\mathbb{R}$. Fix $\alpha > 0$, and let $E_\alpha = \{x \in E : M(f, x) > \alpha\}$. Then the outer measure of $E_\alpha$ satisfies

$$m^*(E_\alpha) \leq \frac{3}{\alpha} \cdot \int_\mathbb{R} f \, dm.$$

**Proof.** Given $x \in E_\alpha$, there is an interval $I_x \in \mathcal{I}(x, 1)$ such that

$$\alpha \cdot m(I_x) \leq \int_{I_x} f \, dm.$$

These intervals form a collection $\mathcal{I}$ that cover $E_\alpha$, so by Theorem 20.1, there is a finite disjoint subcollection ${I_1, \cdots, I_n} \subset \mathcal{I}$ such that

$$m^*(E_\alpha) \leq m(\cup_{I \in \mathcal{I}} I) \leq 3 \cdot \sum_{k=1}^n m(I_k) \leq \frac{3}{\alpha} \sum_{k=1}^n \int_{I_k} f \, dm \leq \frac{3}{\alpha} \cdot \int_\mathbb{R} f \, dm.$$

**Proposition 20.3.** Let $E$ be a bounded measurable subset of $\mathbb{R}$, and let $f$ be a nonnegative integrable function that vanishes almost everywhere on $E$. Then $M(f, x) = 0$ for almost all $x \in E$. 
Proof. Given $\alpha > 0$ and an $\varepsilon > 0$, we may fix an open set $U \supseteq E$ so that $\int_U f \, dm < \varepsilon \alpha / 3$. Now

$$E_\alpha := \{ x \in E : M(f, x) > \alpha \} = \{ x \in E : M(f \cdot \chi_U, x) > \alpha \}.$$ 

Therefore, $m^*(E_\alpha) \leq \frac{3}{\alpha} \int_U f \, dm < \varepsilon$. Since $\varepsilon$ is arbitrary, $m^*(E_\alpha) = 0$, and the result follows. \hfill \blacksquare
21. Absolute continuous functions

A function \( f : [a, b] \to \mathbb{R} \) is called of bounded variation if

\[
\|f\|_{BV} = \sup \left\{ \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| : a = x_0 < x_1 < \cdots < x_n = b \right\}
\]

is finite. We say that \( f \) is absolutely continuous if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every partition \( a = x_0 < x_1 < \cdots < x_n = b \) and every subset \( J \subset \{1, \ldots, n\} \)

\[
\sum_{i \in J} |x_{i+1} - x_i| < \delta \implies \sum_{i \in J} |f(x_{i+1}) - f(x_i)| < \varepsilon.
\]

**Lemma 21.1.** Let \( f \in L_1[a, b] \) and \( F(t) = \int_a^t f(s) \, dm(s) \). Then \( F \) is of bounded variation and absolutely continuous. Moreover, \( F(a) = 0 \) and \( \|F\|_{BV} \leq \int |f| \).

**Proof.** For a partition \( a = x_0 < x_1 < \cdots < x_n = b \) and \( J \subset \{1, \ldots, n\} \) and

\[
\varepsilon_i = \frac{F(x_{i+1}) - F(x_i)}{|F(x_{i+1}) - F(x_i)|}
\]

we have

\[
\sum_{i \in J} |F(x_{i+1}) - F(x_i)| = \left| \int (\sum_{i \in J} \varepsilon_i 1_{[x_i, x_{i+1}]}(s)) \, dm \right| \leq \int |f| \, 1_{\cup_{i \in J}[x_i, x_{i+1}]} \, dm.
\]

Thus for \( J = \{1, \ldots, n\} \) we get

\[
\sum_{i=0}^{n-1} |F(x_{i+1}) - F(x_i)| \leq \int |f| \, dm.
\]

The absolute continuity follows from

\[
\lim_{m(A) \to 0} \int_A |f| \, dm = 0.
\]

(see exam).

**Lemma 21.2.** Let \( f : [a, b] \to \mathbb{R} \) be of bounded variation. Then \( f \) is the difference of two monotone functions \( f_1, f_2 \). If in addition \( f \) is absolutely continuous, then \( f_1 \) and \( f_2 \) may be assume absolutely continuous.

\[
\|f\|_{BV} = f_1(b) + f_2(b) - f(a).
\]

**Proof.** For any partition \( \pi \) we define

\[
p(f, \pi) = \sum_{i=0}^{n-1} \max \{f(x_{i+1}) - f(x_i), 0\}
\]
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\[ n(f, \pi) = \sum_{i=0}^{n-1} \max\{-f(x_{i+1}) + f(x_i), 0\} \]

\[ t(f, \pi) = \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|. \]

Then

\[ t(f, \pi) = p(f, \pi) + n(f, \pi) \]

and

\[ f(b) - f(a) = \sum_{i=0}^{n} (f(x_{i+1}) - f(x_i)) = p(f, \pi) - n(f, \pi). \]

This implies

\[ f(b) - f(a) + n(f, \pi) = p(f, \pi). \]

Now, we take the supremum over all partitions and still have

\[ f(b) - f(a) + \sup_{\pi} n(f, \pi) = \sup_{\pi} p(f, \pi). \]

Moreover,

\[
\sup_{\pi} t(f, \pi) = \sup_{\pi} \left[ p(f, \pi) + n(f, \pi) \right] = \sup_{\pi} \left[ 2p(f, \pi) - (f(b) - f(a)) \right] \\
= \sup_{\pi} p(f, \pi) + \sup_{\pi} p(f, \pi) - (f(b) - f(a)) = \sup_{\pi} p(f, \pi) + \sup_{\pi} n(f, \pi).
\]

For \( a \leq x \leq b \) we define

\[ g(x) = \sup_{\pi = \{a = x_0 < ... < x_n = x\}} p(f, \pi) \]

and

\[ h(x) = \sup_{\pi = \{a = x_0 < ... < x_n = x\}} n(f, \pi). \]

Then \( g \) and \( h \) are increasing functions and

\[ f(x) - f(a) + h(x) = g(x). \]

This yields

\[ f(x) = g(x) - h(x) + f(a). \]

Moreover,

\[ \|f\|_{BV} = g(b) - f(a) + h(b). \]

If in addition \( f \) is absolute continuous then it follows by the definition that \( \sum_i |x_{i+1} - x_i| < \delta \) implies

\[
\sup_{i \in I} \sum_{i \in I} |g(x_{i+1}) - g(x_i)| \leq \sup_{\sum_j |y_{j+1} - y_j| < \delta} \sum_j \max\{f(y_{j+1}) - f(y_j), 0\}. 
\]
Thus we can work with same relation between $\varepsilon$ and $\delta$ for $g$ and $h$.

**Theorem 21.3.** Let $F : [a, b] \to \mathbb{R}$ be an absolute continuous function of bounded variation. Then there exists a function $f \in L_1[a, b]$ such that

$$F(t) = F(a) + \int_a^t f(s)ds$$

and $\|f\|_1 = \|F\|_{BV}$.

**Proof.** We may assume $F(a) = 0$. Let $F = F_1 - F_2$ such that $F_1$ and $F_2$ are positive

$$F_1(b) + F_2(b) = \|F\|_{BV}$$

and such that $F_1$, $F_2$ are absolutely continuous. We define the measure on $A_{\mathbb{R}}$.

$$\nu((s, t]) = F_1(t) - F(s).$$

Using the absolute continuity it is not hard to check that

$$\nu((s, t]) = \sum \nu((s_j, t_j])$$

for every disjoint decomposition. Thus $\nu$ extends to a $\sigma$ additive measure on the borel sets which is absolutely continuous with respect to the Lebesgue measure. Thus $\nu$ extends to Lebesgue measurable set. By the Radon-Nikodym theorem we find a measurable function $f_1$ such that

$$F_1((s, t]) = \int_s^t f_1 dm.$$  

Then

$$\int f_1 dm = F_1(b) - F_1(a) = F_1(b).$$

We apply the same argument to $F_2$ and find a positive element $f_2$ such that

$$\int f_2 dm = F_2(b).$$

Thus $f = f_1 - f_2$ satisfies the assertion by Lemma 21.1.