1. Connection to the Riemann integral

**Definition 1.1.** Let \( f \) be bounded. The oscillation of \( f \) at \( x \) is defined by

\[
\omega(x) = \lim_{\delta \to 0} \sup \{|f(z) - f(y)| : |z - x| < \delta, |y - x| < \delta\}
\]

**Remark 1.2.** \( \omega(x) \) well-defined. In particular, \( \omega(x) \leq 2 \sup |f| < \infty \).

**Remark 1.3.** \( f \) continuous at \( x \) \( \iff \) \( \omega(x) = 0 \)

**Remark 1.4.** \( m(\{x : \omega(x) \neq 0\}) = \lim_{n \to \infty} m(\{x : \omega(x) > 1/n\}) \)

**Definition 1.5.** (Darboux Integral)

For a partition \( \pi = \langle a = x_0, x_1, \ldots, x_n = b \rangle > \) we define the upper and the lower sum by

\[
\mathcal{S}(\pi, f) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{[x_{i-1}, x_i]} f
\]

\[
\mathcal{S}(\pi, f) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f
\]

A bounded function \( f \) is called Darboux-integrable if \( \forall \epsilon > 0 \ \exists \ \text{partition} \ \pi \text{ such that} \)

\[
\|\mathcal{S}(\pi, f) - \mathcal{S}(\pi, f)\| < \epsilon
\]

**Remark 1.6.** A function is Darboux-integrable if and only if it is Riemann-integrable (see Royden for a precise definition).

**Proposition 1.7.** A bounded function is Riemann integrable on \([a, b]\) if and only if the set of discontinuity points has measure 0.

**Proof.** ”\( \implies \)” : Let \( f \) be Darboux-integrable. Let \( \gamma > 0 \) and \( \delta > 0 \). We define \( \epsilon = \gamma \delta \). By definition there exists a partition \( \pi = \langle 0 = x_0, x_1, \ldots, x_n = 1 \rangle > \) such that

\[
\|\mathcal{S}(\pi, f) - \mathcal{S}(\pi, f)\| < \epsilon.
\]

Consider \( x \in (x_i, x_{i+1}) \) such that \( \omega(x) \geq \gamma \). Then

\[
\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f \geq \gamma.
\]

Observe that points with ‘large’ (and hence certain discontinuity) points satisfy

\[
\{x : \omega(x) \geq \gamma\} = \bigcup_{i, \sup f - \inf f \geq \gamma} (x_i, x_{i+1}) \cup \{x_0, \ldots, x_n\}
\]
Hence,
\[
m(x : \omega(x) \geq \gamma) \leq \sum_{i} |x_{i+1} - x_{i}|
\]
\[
\leq \frac{1}{\gamma} \sum_{i} |x_{i+1} - x_{i}|(\sup_{[x_{i}, x_{i+1}]} f - \inf_{[x_{i}, x_{i+1}]} f)
\]
\[
\leq \frac{1}{\gamma} (\mathcal{S}(\pi, f) - \mathcal{S}(\pi, f))
\]
\[
\leq \frac{\epsilon}{\gamma} = \delta
\]

Since \(\delta\) is arbitrary, we get \(m(x : \omega(x) \geq \gamma) = 0\). However, \(\gamma > 0\) is arbitrary and thus Remark (modcont 1.4) implies \(m(x : \omega(x) > 0) = 0\). This means the set of discontinuity points has measure 0.

\(\Longleftarrow\) Define \(\omega_\delta(x) = \sup\{|f(z) - f(y)| : |z - x| < \delta, |y - x| < \delta\}\).

By assumption we have \(m(x : \omega(x) > 0) = 0\). This implies
\[
m(x : \omega(x) \geq \gamma) = 0
\]

for all \(\gamma > 0\). Therefore, we deduce from the monotonicity of \(\omega_\delta(x)\) that

\[
\text{dd} \quad (1.1) \quad m(x : \omega(x) \geq \gamma) = m(x : \lim_{\delta \to 0} \omega_\delta(x) \geq \gamma) = \lim_{\delta \to 0} m(x : \omega_\delta(x) \geq \gamma) = 0
\]

Let \(\gamma > 0\) and \(\epsilon = \frac{\gamma}{2 \sup |f|}\). By (dd 1.1) we deduce the existence of some \(k\) such that
\[
m(x : \omega_{1/k}(x) \geq \gamma) < \epsilon.
\]

Choose \(m > k\) and \(\pi = x_0, x_1, \ldots, x_m > 0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m}\).

Define
\[
S = \{j \in \{1, \ldots, m\} : \exists x \in [x_{j-1}, x_j] \omega_{1/k}(x) \geq \gamma\}.
\]

Then we get
\[
\sum_{j \in S} (\sup_{[x_{i}, x_{i+1}]} f - \inf_{[x_{i}, x_{i+1}]} f)(x_{j-1} - x_{j}) \leq \gamma \sum_{j \in S} |x_{j+1} - x_{j}| \leq \gamma (b - a).
\]

If \(j \notin S\), then
\[
[x_{j-1} - x_{j}] \subset \{x : \omega_{1/k}(x) > \gamma\}.
\]

This implies
\[
\sum_{j \notin S} (\sup_{[x_{i}, x_{i+1}]} f - \inf_{[x_{i}, x_{i+1}]} f)(x_{j-1} - x_{j}) \leq 2 \sup_{[0,1]} |f| m(\bigcup_{j \notin S} [x_{j-1}, x_{j}])
\]
\[
\leq 2 \sup_{[a,b]} |f| m(x : \omega_{1/k}(x) > \gamma) \leq 2 \sup_{[a,b]} |f| \epsilon \leq \gamma.
\]
Putting the pieces together, we find
\[ \sum_j \left( \sup_{[x_j, x_{j+1}]} f - \inf_{[x_i, x_{i+1}]} f \right) (x_{j+1} - x_j) \leq \gamma (b - a) + \gamma = \gamma (1 + b - a). \]

Since \( \gamma > 0 \) is arbitrary we deduce that \( f \) is Darboux integrable.