Let $1 \leq p < \infty$ and $f$ a measurable bounded function such that $f \in L_p$ show that for all $p < q < \infty$ we have $f \in L_q$ and

$$\lim_{q \to \infty} \|f\|_q = \|f\|_\infty.$$ 

**Solution:** Let $p < q < \infty$ and $c = \|f\|_\infty$. Note that $|f(x)| \leq c$ holds a.e.

$$\int |f(x)|^q dm \leq \int |f(x)|^p |f(x)|^{q-p} dm \leq c^{q-p} \int |f(x)|^p dm .$$

Thus $f \in L_q(\mathbb{R})$. For the second part, we consider a natural number $m > c$. Let $n \in \mathbb{N}$ and define the simple function

$$h_l = \sum_{k=0}^{nm} \frac{k}{n} 1_{\frac{k}{n} < |f| \leq \frac{k+1}{n}}$$

and

$$h_u = \sum_{k=0}^{nm} \frac{k+1}{n} 1_{\frac{k}{n} < |f| \leq \frac{k+1}{n}} .$$

Note that $h_l \leq |f| \leq h_u$ and $f \in L_q$ implies that with Chebychev that $m(\frac{k}{n} < |f| \leq \frac{k+1}{n}) < \infty$. By a previous hw problem we get for $h \in \{h_l, h_u\}$ and $a_k = k/n$ or $k+1/n$ that

$$\lim_{p \to \infty} \|h\|_p = \lim\left( \sum_k a_k^p m(\frac{k}{n} < |f| \leq \frac{k+1}{n}) \right)^{\frac{1}{p}} = \sup_{k: m(\frac{k}{n} < |f| \leq \frac{k+1}{n}) \neq 0} a_k .$$

Let $k_c$ be such that $k_c < nc \leq k_c + 1$. Thus we get

$$\lim_{p} \|h_u\|_p = k_c + 1/n$$

and

$$\lim_{p} \|h_l\|_p = k_c$$

This yields

$$\limsup_{p} \|f\|_p \leq \lim_{p} \|h_u\|_p \leq k_c + 1/n \leq c + \frac{1}{n}$$

and

$$\liminf_{p} \|f\|_p \geq \lim_{p} \|h_l\|_p \geq k_c/n \geq c - \frac{1}{n} .$$

Letting $n \to \infty$ we deduce the assertion. 

Show directly that for $1 \leq p < \infty$ we have

$$\ell_p^* = \ell_p$$

and that $\ell_\infty^* \neq \ell_1$. 

1
Solution: Since we proved Holder’s inequality for arbitrary measure space, we know that
\[
|\sum_n a_n b_n| \leq \left(\sum_n |a_n|^p\right)^{1/p} \left(\sum_n |b_n|^q\right)^{1/q}
\]
whenever \(1/p + 1/q = 1\), i.e. \(q = p'\). Thus the mapping \(u : \ell_p' \to \ell_p^*\) defined by \(u(b_n)((a_n)) = \sum_n a_n b_n\) is satisfies
\[
\|u(b_n)\|_{\ell_p^*} \leq \left(\sum_n |b_n|^q\right)^{1/q}.
\]
We will now show that \(u\) is surjective. Indeed, let \(\phi : \ell_p \to \mathbb{R}\) be a linear continuous map such that
\[
|\phi(a_n)| \leq \left(\sum_n |a_n|^p\right)^{1/p}
\]
We define
\[
b_n = \phi(e_n)
\]
where \(e_n\) is the \(n\)-unit vector. Let \(m \in \mathbb{N}\). From the equality consideration for the Holder inequality, we deduce
\[
(\sum_{n \leq m} |b_n|^q)^{1/m} \leq \sup_{n \leq m} \|a_n\| \leq \sum_{n \leq m} |a_n|b_n| \leq \|\phi\|.
\]
Taking the sup over \(m\) we get
\[
(\sum_n |b_n|^q)^{1/q} \leq \|\phi\|.
\]
Since \(1 \leq p < \infty\), we know that simple functions are dense. However, simple function here correspond to finite sequences. By the unique extension principle we deduce that the sequence \((b_n)\) defined above satisfies \(u(b_n) = \phi\) and
\[
(\sum_n |b_n|^q)^{1/q} \leq \|u(b_n)\|_{\ell_p^*}.
\]
I will assume some knowledge in logic for proving \(\ell^*_\infty \neq \ell_1\). (The argument breaks down because simple function=finite sequences are no longer dense). Let \(\mathcal{U}\) be a free ultrafilter over \(\mathbb{N}\). Then we define
\[
\phi((a_n)) = \lim_{n \in \mathcal{U}} a_n
\]
One can show that for every compact set and every ultrafilter \(\mathcal{U}\) the limit with respect to \(\mathcal{U}\) exists. Here we may consider \((a_n) \subset [-c, c]\) and the ask \(A_\infty \{n \in \mathbb{N} :\)
\[-c \leq a_n \leq 0 \} \in \mathcal{U}\) or \(A^c \in \mathcal{U}\). The we split \([-c,0]\) and \((0,c]\) in two intervals and continue. It then easily follows that \(\phi\) is well-defined and satisfies

\[
|\phi((a_n)| \leq \sup_n |a_n| .
\]

However, \(\phi\) does not come from an element in \(\ell_1\). Indeed, for every \((b_n) \in \ell_1\) and every infinite set \(A\) we have

\[
c_k = u((b_n))(1_{A \cap (k,\infty)}) = \sum_{n \in A, n \geq k} b_k
\]

and \(\sum_k |b_k|\) finite implies \(\lim_k c_k = 0\). However, let \(A \in \mathcal{U}\) be an infinite subset. Then

\[
\phi(1_{A \cap [k,\infty)}) = 1
\]

holds for all \(k\).

Problem 7a) and problem 7b) on page 104.

7a) If \(f\) is a monotone increasing function, then

\[
\lim_{s \uparrow t} f(s) = \sup_{s < t} f(s)
\]

and

\[
\lim_{s \downarrow t} f(s) = \inf_{s > t} f(s) .
\]

Thus these limits exist. A similar argument applies for monotone deceasing function. Thus for a function of bounded variation \(g\) we deduce the result by writing \(g = f_1 - f_2\) with \(f_i\) increasing.

Consider again \(f\) monotone increasing on \([a,b]\). Let \(A\) be the set of continuity points. For fixed \(n \in \mathbb{N}\) we consider

\[
A_n = \{ t \in (a,b) : \lim_{s \uparrow t} f(s) + \frac{1}{n} \leq \lim_{s \downarrow t} f(s)\}
\]

Let \(t_1, \ldots, t_m\) be \(m\) distinct elements. We may assume \(a \leq t_1 < t_2 < \cdots < t_m \leq b\). We choose points \(a < s_1 < t_1 < s_2 < t_2 < s_3 < \cdots < t_m < s_m < b\). Then

\[
f(b) - f(a) = f(b) - f(s_m) + f(s_m) - f(s_{m-1}) + \cdots + f(s_2) - f(s_1) + f(s_1) - f(a) \geq \frac{m}{n} .
\]

Thus \(A_n\) has at most \(n(f(b) - f(a))\) many elements. Since \(\bigcup_n A_n\) is the collection of all discontinuity points in \((a,b)\) we are done.

7b) Let \((r_n) \subset [0,1]\) be an enumeration of the rational points and \((a_n)\) be a positive numbers sequence such that \(\sum_n a_n = 1\). We define

\[
f = \sum_n a_n 1_{[r_n, r_{n+1}]}
\]
The $f$ is obviously monotone and has jumps at all points $r_n$, i.e. \( \lim_{t \to r_n, t < r_n} f(t) + a_n = \lim_{t \to r_n} f(t) \). Now, let $t$ be an irrational point. let $\varepsilon > 0$ and $n_0$ such that $\sum_{n > n_0} a_n < \varepsilon$. Let $\delta = \min_{j=1, \ldots, n_0} |t - r_j|$. For every $|t - s| < \delta$ we have

$$|f(t) - f(s)| \leq \sum_{n > n_0} |a_n| < \varepsilon.$$ 

That’s it. \( \blacksquare \)

Problem 10a) and problem 10b) on page 104.

10a) $g(x) = x^2 \cos(x^{-2})$. Let $n \in \mathbb{N}$ and define $s_{n-j}$ such that $1/s_{n-j}^2 = \pi/2 + \pi j$ for $j = 0, \ldots, n$. Then we have

$$\sum_{j=0}^{n-1} |g(s_j) - g(s_{j+1})| = \sum_{j=0}^{n-1} |s_{n-j}^2 + s_{n-j-1}^2| = \sum_{j=1}^{n} |s_j^2 + s_{j-1}^2| \geq \sum_{j=1}^{n} 1/(\pi/2 + \pi j) \geq \frac{1}{\pi} \sum_{j=1}^{n} 1/j.$$

Since $\sum_{j} 1/j = \infty$ we deduce the assertion.

For b) and $g(x) = x^2 \sin(1/x)$ we note that it suffices to show that $g'$ is in $L_1$. Except for 0 we have $g'(x) = 2x \sin(1/x) + x^2 \cos(1/x)(-x^{-2}) = 2x \sin(1/x) - \cos(1/x)$. Thus $g'$ is almost everywhere bounded and thus in $L_1$. Hence $g$ is of bounded variation.

Problem 16) p111)-If time permits I will explain this problem Friday in class. No I didn’t—but here is the solution.

a) Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function. Then we have

$$f(a) + \int_a^x f' \, dm \leq f(x)$$

for every $x \in [a, b]$. Thus we may define $g(x) = f(a) + \int_0^x f' \, dm$ and $h(x) = f(x) - g(x)$. By the fundamental theorem (Lebesgue differentiation theorem) we have $h'(x) = f'(x) - g'(x) = 0$ almost everywhere. Moreover, let $y < x$ then

$$h(x) - h(y) = f(y) - f(x) - \int_y^x f' \, dm \geq 0$$

be the differentiation theorem for monotone functions. Thus $h$ is monotone and $h' = 0$ a.e.-i.e. $h'$ is singular.

In b) and c) we prove the following
Lemma 0.1. Let $f$ be a monotone function. $f$ is singular if and only if for every $a \leq x \leq b$ and $\varepsilon > 0$ and $\delta > 0$ there exists non-overlapping intervals $[y_j, y_j + d_j]$ such that

$$\sum_j d_j < \varepsilon \quad \text{and} \quad f(x) - f(a) < \sum_j (f(y_j + d_j) - f(y_j)) + \delta.$$ 

Proof. $"\Rightarrow":$ Let $f$ be singular. Let $\varepsilon > 0$ and $E = \{x : f'(x) = 0\} \cap [a+\varepsilon, b-\varepsilon]$. We may find an open subset $0$ of $(a, b)$ such that $E \subset O$ and $m(O) < m(E) + \varepsilon$. For every $x \in E$ and $\gamma > 0$ we may find $0 < h < \gamma$ such that $f(x + h) - f(x) < \gamma h$. By the Vitali covering lemma we obtain non-overlapping intervals $([x_k, x_k + h_k])_{k=1,\ldots,m}$ such that

$$\sum_{k=1}^m h_k > m(E \cap [a + \varepsilon, b - \varepsilon]) - \varepsilon \geq b - a - 3\varepsilon.$$ 

Without loss of generality we may assume $a + \varepsilon < x_1 < x_1 + h_1 < x_2 < x_2 + h_2 < \ldots < x_m < x_m + h_m \leq b$. We define $y_0 = a$ and $d_1 = x_1 - y_1, y_j = x_j + h_j$ and $d_j = x_{j+1} - y_j$. Finally $d_m = b - y_m$. Then

$$b - a = \sum_{k=1}^m d_k + \sum_{j=0}^m d_j \geq (b - a) - 3\varepsilon + \sum_{j=0}^m d_j.$$ 

This yields $\sum_j d_j < 3\varepsilon$. On the other hand

$$f(b) - f(a) = \sum_k (f(x_k + h_k) - f(x_k)) + \sum_j (f(y_j + d_j) - f(y_j))$$

$$\leq \sum_k \gamma h_k + \sum_j (f(y_j + d_j) - f(y_j))$$

$$\leq \gamma (b - a) + \sum_j (f(y_j + d_j) - f(y_j)).$$

This is exactly what we want to prove for $x = b$. However, since $f'$ also holds on $[a, x]$ we are done.

$"\Leftarrow":$ Let $f = g + h$ such that $h$ is singular and $g$ is absolutely continuous and $g(a) = f(a)$. Let $a \leq x \leq b$ we want to show $g(x) = g(a)$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\sum_k h_k < \delta \Rightarrow \sum_j (g(x_k + h_k) - g(x_k)) < \varepsilon.$$
(g is absolutely continuous). By assumption we find non-overlapping interval \([y_j, y_j + d_j]\) such that \(\sum_j d_j < \delta\) and

\[
f(x) - f(a) < \sum_j [f(y_j + d_j) - f(y_j)] + \varepsilon.
\]

Let \(x_k, h_k\) such that \(\bigcup_k [x_k, x_k + h_k] \cup \bigcup_j [y_j, y_j + h_j] = [a, b]\) and the two unions only overlap in the endpoints. Then we get

\[
f(x) - f(a) < \varepsilon + \sum_j [h(y_j + d_j) - h(y_j)] + \sum_j [g(y_j + d_j) - g(y_j)]
\]

\[
2\varepsilon + \sum_j [h(y_j + d_j) - h(y_j)]
\]

\[
\leq 2\varepsilon + \sum_j [h(y_j + d_j) - h(y_j)] + \sum_k [h(x_k + h_k) - h(x_k)]
\]

\[
= 2\varepsilon + h(x) - h(a) = 2\varepsilon + h(x).
\]

Since \(\varepsilon > 0\) is arbitrary we get \(f(x) = h(x) + f(a)\). Thus \(f\) is singular.

**Proposition 0.2.** Let \((f_n)\) positive singular monotone functions such that \(f(x) = \sum_n f_n\) converges point-wise. Then \(f\) is singular.

**Proof.** Without loss of generatily we may assume \(f_k(a) = 0\) for all \(k \in \mathbb{N}\). Let \(a \leq x \leq b, \varepsilon > 0, \delta > 0\). Let \(n_0\) be such that \(\sum_{k=1}^{n_0} f_k(x) > f(x) - \delta\). Obviously, \((\sum_{k=1}^{n_0} f_k)'(x) = \sum_{k=1}^{n_0} f_k'(x) = 0\) holds a.e. Thus we find non-overlapping intervals \(([y_j, y_j + d_j])\) such that \(F_{n_0} = \sum_{k=1}^{n_0} f_k\) satisfies \(\sum_j d_j < \delta\) and

\[
F_{n_0}(x) - \varepsilon < \sum_j [F_{n_0}(y_j + d_j) - F_{n_0}(y_j)].
\]

This implies by monotonicity of the \(f_n\)’s and by point-wise convergent that

\[
f(x) - 2\varepsilon = F_{n_0}(x) < \sum_j [F_{n_0}(y_j + d_j) - F_{n_0}(y_j)]
\]

\[
= \sum_j \sum_{n=1}^{n_0} [f_n(y_j + d_j) - f(y_j)]
\]

\[
= \sum_j \sum_{n=1}^{\infty} [f_n(y_j + d_j) - f(y_j)]
\]

\[
= \sum_j [f(y_j + d_j) - f(y_j)].
\]

Thus \(f\) is also singular-i.e. a function which creates everything out of nothing.
e) Finally consider \((r_n)\) an enumeration of the rationals and \((a_n)\) strictly positive such that \(\sum_n a_n < \infty\). Then

\[
f = \sum_n a_n 1_{[r_n, 1]}
\]

is singular by our previous proposition. Moreover, \(f\) is strictly increasing because between two points \(x < y\) we find \(x < r_n < y\) and hence \(f(y) - f(x) > a_n\). ■

Let \(\mu << \nu\) be finite probability measures. Let \(f_1\) and \(f_2\) be Radon Nikodym derivatives such that

\[
\mu(E) = \int_E f_1 d\nu \quad \text{and} \quad \mu(E) = \int_E f_2 d\nu
\]

What can you say about \(f_1\) and \(f_2\). In which sense is the Radon-Nikodym derivative unique.

**Solution:** Consider \(E_n = \{\omega \in \Omega : f_1(\omega) > f_2(\omega) + \frac{1}{n}\}\). Then

\[
0 = \int_{E_n} (f_1 - f_2) d\nu \geq \frac{1}{n} \nu(E_n) \geq 0.
\]

Thus \(\nu(E_n) = 0\). This implies \(f_1 = f_2\) holds \(\nu\) almost everywhere. Hence the Radon-Nikodym derivative is uniquely determined up to set of measure 0 for \(\nu\). ■

Let \(\Omega = \{1, \ldots, n\}\) and \(\nu\) the counting measure \(\nu(A) = |A|\). Let \(\mu\) be an arbitrary measure calculate the Radon-Nikodym derivative.

**Solution:** We consider the positive numbers

\[
\mu_i = \mu(\{i\}).
\]

Define \(f(i) = \mu_i\). Then

\[
\mu(E) = \sum_{i \in E} \mu_i = \int_E f d\nu
\]

holds for every set. ■