Real Analysis-Homework 8

Due date: Monday, November 1

(1) (15P) Let $\Omega = \mathbb{R}$, $\Sigma = \{ A : A \text{countable or } A^c \text{countable} \}$ and
$$
\mu(A) = \begin{cases} 
\infty & A^c \text{is countable} \\
0 & A \text{is countable}
\end{cases}.
$$

Show that $\mu$ is $\sigma$-additive. Consider $f = 1_{\mathbb{R}}$. Show that
$$
I(f) = 0 \quad \mu(\{ x \in \mathbb{R} : f(x) \geq \frac{1}{2} \}) = \infty.
$$

(2) Let $(\Omega, \Sigma, \mu)$ be a measure space (not necessarily $\sigma$-finite). We will now say that a function $f : \Omega \to [-\infty, \infty]$ is measurable in the strong sense if there exists a set $F \in \Sigma$ of measure 0 and a sequence $(g_n)$ of simple functions such that
$$
f(\omega) = \lim_n g_n(\omega)
$$
holds for all $\omega \in F^c$.

(a) (15P) Let $f \geq 0$ and measurable in the strong sense. Show that for every $\lambda > 0$ the set $E_\lambda = \{ \omega \in \Omega : f(\omega) \geq \lambda \}$ is $\sigma$-finite, i.e. there exists $G_n \in \Sigma$ with finite measure such that $E_\lambda = \bigcup_n G_n$. (Hint: use the functions $h_n = \inf_{m \geq n} g_n$.

(b) (10P) Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Show that every measurable function is measurable in the strong sense. (Hint: use the functions $f_\varepsilon$ below).

(3) (20P) Let $(\Omega, \Sigma, \mu)$ be a measure space such that $\mu(\Omega) < \infty$. Let $f \geq 0$ and $\varepsilon > 0$. Consider
$$
f_\varepsilon = \sum_{k=0}^{\infty} (k\varepsilon)1_{\{k\varepsilon \leq f < (k+1)\varepsilon\}}.
$$

Show that
$$
I(f) = \lim_{\varepsilon \to 0} I(f_\varepsilon).
$$

Conclude that
$$
I(f) = \inf \{ \sum_k r_k \mu(E_k) : f \leq \sum_k r_k 1_{E_k} \}.
$$

(4) In this exercise we want to establish the link between areas and integrals. Let $(\Omega, \Sigma, \mu)$ be a finite measure space ($\mu(\Omega) < \infty$). On $\tilde{\Omega} = [0, \infty) \times \Omega$ we consider the algebra $A$ generated by the sets $[s, t] \times E$, $0 \leq s \leq t \leq \infty$, $E \in \Sigma$. 

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(a) (10P) Show that every element $F$ in $A$ can be written as
\[ F = \bigcup_{k=1}^{m} G_k \times E_k \]
such that the $E_k$’s are disjoint and $G_k \in \mathbb{R}$. We then define
\[ \nu(F) = \sum_{k} m(G_k)\mu(E_k) . \]
It can be shown that for every $F \in A$ and for every disjoint union $F = \bigcup_j F_j$ of disjoint sets in $A$ we have
\[ \nu(F) = \sum_j \nu(F_j) . \]
Therefore we may extend $\nu$ to a $\sigma$-additive measure on the $\sigma$-algebras $\tilde{\Sigma}$ of measurable subsets (in the Caratheodory sense) of $[0, \infty] \times \Omega$.

(b) (5P) Let $E \in \Sigma$ be a set of measure 0. Show that
\[ \nu([0, \infty) \times E) = 0 . \]

(c) (15P) Let $f : \Omega \to [0, \infty]$ be measurable such that $I(f) < \infty$. Show that the graph of $f$
\[ G(f) = \{(r, \omega) : 0 \leq r \leq f(\omega)\} \]
has finite measure with respect to $\nu$ and satisfies $\nu(G(f)) \leq I(f)$.
(Hint for a simple function $h$ we have $\nu(G(h)) = I(h)$.

(d) (10P) Let $f : \Omega \to [0, \infty]$ be measurable and assume
\[ \nu(G(f)) < \infty . \]
Show that $I(f) \leq \nu(G(f))$. (Hint: consider a simple function $0 \leq h \leq f$).