4. Differentiation of monotone functions

DEFINITION 4.1. *A borel measure on* $(-a,b]$ *is a* σ*-additive measure defined on the borel algebra of* $(a,b]$*, i.e. the smallest* σ*-algebra containing all the open subset of* $(a,b]$.

PROPOSITION 4.2. *Let* μ *be a finite borel measure (i.e.* $\mu((a,b]) < \infty$*. Then*

$$F(t) \;=\; \mu((a,t])$$

is an increasing function such that

$$\lim_{s\downarrow t} F(s) \;=\; F(t)\,.$$

Conversely, every such function defines a uniquely determines a borel measure on $(a,b]$.

PROOF. Let μ be a finite borel measure. By σ-continuity we have

$$\lim_{n\to\infty} \mu(\bigcup_n (a,b+s_n])\mu((a,b])$$

for every decreasing sequence (s_n) with $\lim_n s_n = 0$. For the converse we use the extension procedure. Assume that F is as above. We consider the algebra $A_{\mathbb{R}}$ generated by the intervals $(s,t]$. For such an interval we define

$$\mu_F((s,t]) \;=\; F(t) - F(s)\,.$$

We have have to show that

$$\mu_F((s,t]) \;\le\; \sum_j \mu(B_j)$$

whenever $B_j \in A$ and $\bigcup_j B_j = (s,t]$ is a disjoint union. Since every B_j is a finite union of intervals we may as well assume as disjoint partition

$$(s,t] \;=\; \dot{\bigcup}_j (s_j,t_j]\,.$$

Let $\varepsilon > 0$. For fixed j we may chose $\delta_j > 0$ such that

$$F(t_j+\delta_j) - F(s_j) < F(t_j) - F(s_j) + 2^{-j}\varepsilon\,.$$

Let $\delta > 0$. Then

$$[s+\delta,t] \subset \bigcup_j (s_j, t_j+\delta_j)\,.$$

By compactness we can find n_0 such that

$$[s+\delta, t] \subset \bigcup_{j=1}^{n} (s_j, t_j + \delta_j) \, .$$

Now, we rearrange the intervals such that $s_1 < s+\delta < t_1+\delta_1$ and $s_2 < t_1+\delta_1 < t_2+\delta_2$, etc such that finally $t < t_m + \delta_m$. This gives

$$\begin{aligned} F(t) - F(s+\delta) \leq \sum_{i=1}^{m} (F(t_i + \delta_i) - F(s_i)) \;&\leq\; \sum_j \mu_F((s_j, t_j]) + 2^{-j}\varepsilon \\ &\leq \varepsilon + \sum_j \mu_F((s_j, t_j]) \, . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$F(t) - F(s+\delta) \;\leq\; \sum_j \mu_F((s_j, t_j]) \, .$$

Using $\lim_{\delta \to 0} F(s+\delta) = F(s)$, we deduce

$$F(t) - F(s) \;\leq\; \sum_j \mu_F((s_j, t_j]) \, .$$

Applying the Caratheodory extension procedure we find a σ-additive measure $\tilde{\mu}_F$ which satisfies

$$\tilde{\mu}_F((s,t]) \;=\; F(t) - F(s) \, .$$

Note that sets of measure 0 are defined by the outer measure μ_F^* which is uniquely determined by the values $\tilde{\mu}_F((s,t])$. Since we are dealing with finite measure, we see that for every measurable set $E \subset (a,b]$ we may find $B \in A$ such that

$$\mu_F^*(E \Delta B) < \varepsilon \, .$$

This implies

$$\mu_F(B) - \varepsilon < \tilde{\mu}_F(E) < \mu_F(B) + \varepsilon \, .$$

Thus the extension is uniquely determined from its values on intervals $(s,t]$. ■

In the following we assume that $F : [a,b] \to \mathbb{R}$ is a monotone increasing function, continuous from the right. We need the following four derivatives

$$\begin{aligned} D^+F(x) &= \limsup_{h \downarrow 0} \frac{F(x+h) - F(x)}{h} \, , \\ D^-F(x) &= \limsup_{h \downarrow 0} \frac{F(x) - F(x-h)}{h} \, , \\ D_+F(x) &= \liminf_{h \downarrow 0} \frac{F(x+h) - F(x)}{h} \, , \end{aligned}$$

$$D_-F(x) \;=\; \liminf_{h\downarrow 0} \frac{F(x)-F(x-h)}{h}\,.$$

Note that

$$D^+F(x) \;\geq\; D_+F(x) \quad \text{and} \quad D^-F(x) \;\geq\; D_-F(x)\,.$$

We recall that F is differentiable at x if all the values coincide and are finite. The following result is proved following almost verbatim Theorem 5.3 in Royden (see page 100).

THEOREM 4.3. *Let F be a monotone increasing function. Then F is differentiable almost everywhere. Moreover,*

$$\int_a^b F'(x)dx \;\leq\; F(b)-F(a)\,.$$

PROOF. In this proof we define $F(x) = F(b)$ for $x \geq b$. As an example we consider

$$E_{u,v} \;=\; \{x \,:\, D^+F(x) > u > v > D_-F(x)\}$$

for all rational u, v. Our aim is to show that $m*(E_{uv}) = 0$. Then we may conclude that $D_+F(x) \leq D^+F(x) \leq D_-F(x) \leq D^-F(x)$ a.e. (Similarly, one can show show that $D^-F(x) \leq D_+F$ and thus F is differentiable almost everywhere.)

Now, we consider $s = m^*(E_{uv})$ and $\varepsilon > 0$. We may find an open set O such that $E_{u,v} \subset O$ and $m(O) < s+\varepsilon$. For each point $x \in E_{uv}$ there is a small interval $[x-h, x] \subset O$ such that

$$F(x)-F(x-h) < vh\,.$$

By Theorem **??**, we may find a finite disjoint collection $I_1, ..., I_N$ of such intervals such that

$$m(\bigcup_{n=1}^N \text{Interior}(I_n)) > s-\varepsilon\,.$$

We write $I_n = [x_n - h_n, x_n]$ and get from disjointness that

$$\sum_{n=1}^N (F(x_n)-F(x_n-h_n)) < v\sum_n h_n < vm(O) < v(s+\varepsilon)\,.$$

We define $A = \bigcup_{n=1}^N \text{Interior}(I_n) \cap E_{u,v}$. Every $y \in A$ is the left endpoint of an interval $(y, y+k)$ contained on some I_n such that

$$F(y+k)-F(k) > uk\,.$$

We apply the covering Theorem **??** again and get disjoint intervals $J_1, ..., J_M$ such that $\bigcup_i J_i$ contains a subset of A of measure $> s - 2\varepsilon$. Then

$$\sum_{i=1}^{M} [F(y_k + k_i) - F(k_i)] > u \sum_i k_i > u(s - 2\varepsilon) .$$

Note that every J_i is contained in some I_n. By monotonicity and disjointness we get

$$\sum_{J_i \subset I_n} [F(y_k + k_i) - F(k_i)] \ \leq\ F(x_n) - F(x_n - h_n) .$$

Thus we have

$$u(s - 2\varepsilon) < \sum_{i=1}^{M} [F(y_k + k_i) - F(k_i)] \ \leq\ \sum_{n=1}^{N} (F(x_n) - F(x_n - h_n)) \ \leq\ v(s + \varepsilon) .$$

Passing to the limit $\varepsilon \to 0$ we get $us \leq vs$. Since $u > v$ we must have $s = 0$.
In the following we may assume that

$$g(x) \ =\ \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} \ =\ F'(x)$$

exists almost everywhere. We define

$$f_n(x) \ =\ n\left(F(x + \frac{1}{n}) - F(x) \right) .$$

Since F is increasing we know that $f_n(x) \geq 0$. Since $\lim_n f_n(x) = F'(x)$ a.e. we know that F' is measurable. By Fatou's Lemma we find (with the use of a practice problem)

$$\begin{aligned}
\int_a^b f &\leq \liminf_n n \int_a^b [F(x + \frac{1}{n}) - F(x)] dx \\
&= \liminf_n \left(n \int_b^{b+\frac{1}{n}} F - n \int_a^{a+\frac{1}{n}} F \right) \\
&= \liminf_n \left(F(b) - n \int_a^{a+\frac{1}{n}} F \right) \\
&= F(b) - \lim_n n \int_a^{a+\frac{1}{n}} F \ =\ F(b) - F(a) . \qquad \blacksquare
\end{aligned}$$

COROLLARY 4.4. *A function of bounded variation is differentiable almost everywhere.*

COROLLARY 4.5. *Let μ be a finite Borel measure on $(a, b]$. Then there exists an absolute continuous measure μ_n and singular measure μ_s (see below for the definition) such that*

$$\mu \;=\; \mu_n + \mu_s \, .$$

PROOF. Consider $F(x) \;=\; \mu((a, x])$. Then F is differentiable a.e. and we may define the absolute continuous measure

$$\mu_n((a, x]) \;=\; \int_a^x F'(x)dm \, .$$

Then $G(x) \;=\; F(x) - \int_a^x F'(x)dm$ is again an increasing function. The singular measure is determined by

$$\mu_s((a, x]) \;=\; G(x) \, .$$

Obviously, $G' = 0$ a.e. That is the definition of singular (see Problem 16 on page 111) in Royden for more information). ■