1. Some remarks on the ternary Cantor function

The complement of the Cantor set in $[0, 1]$ is given by

$$O = \bigcup_{n=0}^{\infty} \bigcup_{a \in \{0, 2\}^n} O_a$$

where for $n = 0$, $a = \emptyset$

$$O_a = \left( \frac{1}{3}, \frac{2}{3} \right)$$

and for $n \geq 0$, $a = (a_1, ..., a_n)$ we have

$$O_a = \left( \sum_{i=1}^{n} a_i 3^{-i} + 3^{-(n+1)}, \sum_{i=1}^{n} a_i 3^{-i} + 23^{-(n+1)} \right).$$

On such a set $O_a$ we define

$$f(x) = \sum_{i=1}^{n} \frac{a_i}{2} 2^{-i} + 2^{-(n+1)}$$

For $n = 0$ we use the value $\frac{1}{2}$.

**Lemma 1.1.** $f$ is uniformly continuous.

**Proof.** Let us consider $x_1$ and $x_2$ in $O$ such that $|x_1 - x_2| < 3^{-n}$. We may write

$$x_1 = \sum_{i=1}^{\infty} a_i 3^{-i}$$

and

$$x_2 = \sum_{i=1}^{\infty} b_i 3^{-i}$$

such that there exists a smallest integer $k$ and $m$ with $a_k = 1$, $b_m = 1$. Let $j$ be the smallest integer such that $a_j \neq b_j$.

**Case 1)** $j < \min(k, m)$: W.l.o.g. we may assume $a_j = 2$ and $b_j = 0$. Then

$$3^{-n} \geq x_1 - x_2 \geq 23^{-j} - 3^{-j} = 3^{-j}.$$ 

This yields $j \geq n$ and

$$|f(x_1) - f(x_2)| \leq \sum_{i \leq j} |a_i - b_i|2^{-i} \leq 22^{-j} \leq 42^{-n}.$$ 

**Case 2)** $j \geq \min(k, m)$. If $k = m$, then $f(x_1) = f(x_2)$ are we are fine. Let us assume $k > m$. Then $a_m \in \{0, 2\}$ and $b_m = 1$. Thus $j = m$. 

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**Case 2a)** $a_m = 2$: Let us fix the smallest $j > k$ such that $a_j \neq 0$. We get
\[ f(x_1) - f(x_2) = \sum_{i=m+1}^{k-1} \frac{a_i}{2} 2^{-i} + 2^{-m}. \]
Moreover,
\[ 3^{-n}x_1 - x_2 \geq 3^{-k} + a_j 3^{-j} - \left( \sum_{i=k+1}^{\infty} b_i 3^{-i} \right) \geq a_j 3^{-j}. \]
Thus $j \geq n$ and hence
\[ |f(x_1) - f(x_2)| \leq \sum_{i\geq j} \frac{a_i}{2} 2^{-i} + 2^{-m} \leq 42^{-j} \leq 42^{-n}. \]

**Case 2b)** $a_m = 0$: similar.

**Corollary 1.2.** There exists a continuous strictly increasing function $g : [0, 1] \rightarrow [0, 2]$ such that $m(C) = 1$.

**Proof.** Let $F$ be the unique extension of $f$ to $[0, 1]$. One can show that $f$ is increasing and $f(0) = 0, f(1) = 1$. Then $g(x) = F(x) + x$ is strictly increasing and by the intermediate value Theorem we have $g([0, 1]) = [0, 2]$. Moreover, $g(O_n)$ is an interval with length $3^{-(n+1)}$. Thus yields $m(g(O)) = 1$. Taking complements yields the assertion.

**Corollary 1.3.** There exists a Lebesgue measurable set which is not borel.

**Proof.** Let $D \subset m(C)$ be a non-measurable set. Assume that $B = g^{-1}(D) \subset C$ is a borel set. Then we deduce that
\[ D = g(B) = (g^{-1})^{-1}(B) \]
is also a borel set because $g^{-1}$ is continuous (why). This contradiction shows that $B$ is not a borel set. However, $B \subset C$ is a subset of a set of measure 0 and hence Lebesgue measurable.