4. Banach spaces

**Definition 4.1.** A normed space is given by a vector space $V$ (over $K = \mathbb{R}$ or $K = \mathbb{C}$) and a function $\| \| : V \to [0, \infty)$ satisfying the following conditions

i) $\|x\| = 0 \iff x = 0$,
ii) $\|\lambda x\| = |\lambda| \|x\|$, 
iii) $\|x + y\| \leq \|x\| + \|y\|$, 

for all $x, y \in V$, $\lambda \in K$. The associated metric on $(V, \| \|)$ is defined by 

$$d(\|, \|)(x, y) = \|x - y\|.$$

**Remark 4.2.** $+: V \times V \to V$ given by $+(x, y) = x + y$ and $\cdot: K \times V \to V$ given by $(\lambda, x) = \lambda x$ are continuous. Moreover, $\| \| : V \to [0, \infty)$ is continuous.

In the following we will mostly consider real vector spaces (because the name of our course is real analysis).

**Definition 4.3.** A Banach space is a normed vector space such that $(V, d(\|, \|))$ is complete.

**Example 4.4.**

1. On $V = \mathbb{R}^n$ we define 

$$\|x\|_p = \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}}$$

and $\|x\|_{\infty} = \max_{i=1,\ldots,n} \|x_i\|$. Then $(\mathbb{R}^n, \| \|_p)$ is Banach space (see below for the triangle inequality).

2. $\ell_p = \{(x_n) : \sum_n |x_n|^p < \infty\}$ is a Banach space with respect to 

$$\|(x_n)\|_p = \left( \sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}}.$$

3. If $\| \|$ is a norm on $\mathbb{R}^n$, then $(\mathbb{R}^n, \| \|)$ is a Banach space.

4. $(C[0,1], \| \|_1)$ where 

$$\|f\|_1 = \int_0^1 |f(s)|ds$$

is a normed space, but not a Banach space.

**Proposition 4.5.** Let $X$ be a normed space and $Y$ be a Banach space. We define $L(X, Y)$ as the space of map $T : X \to Y$ which are linear, i.e. 

$$T(x + \lambda y) = T(x) + \lambda T(y).$$
and continuous. The norm on \(L(X,Y)\) is given by
\[
\|T\|_{op} = \sup_{\|x\| \leq 1} \|T(x)\|.
\]
Then \(L(X,Y)\) is a Banach space.

**Proof.** Let us first show that a linear map \(T : X \to Y\) is continuous iff \(\|T\| < \infty\). Indeed, if \(\|T\|\) is finite, then
\[
\|T(x) - T(y)\| = \|T(x - y)\| \leq \|T\|_{op}\|x - y\|
\]
holds for all \(x, y \in V\). Thus \(T\) is Lipschitz and thus continuous. For the converse, we assume that \(T\) is continuous. Then \(T^{-1}(B(0,1))\) is open and henceforth contains \(B(0, \varepsilon)\) for some \(\varepsilon > 0\). Now let \(\|x\| \leq 1\) and \(0 < \delta < \varepsilon\). Then \(\|(\varepsilon - \delta)x\| < \varepsilon\) and hence
\[
\|T(x)\| = (\varepsilon - \delta)^{-1}\|T(\varepsilon - \delta)(x)\| < (\varepsilon - \delta)^{-1}.
\]
This shows that \(\|T\|_{op} \leq (\varepsilon - \delta)^{-1}\) for every \(\delta > 0\) and thus \(\|T\|_{op} \leq \varepsilon^{-1}\). Now, we observe that \(\|\|_{op}\) is a norm. We only check the triangle inequality. Indeed,
\[
\|T + S\|_{op} = \sup_{\|x\| \leq 1} \|T(x) + S(x)\| \leq \sup_{\|x\| \leq 1} \|T(x)\| + \|S(x)\| \leq \|T\|_{op} + \|S\|_{op}.
\]
Finally we have to show that \(L(X,Y)\) is complete. Let \((T_n)\) be a Cauchy sequence of linear maps. For fixed \(x \in X\), we have
\[
\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\| \|x\|.
\]
Thus \((T_n(x))\) is Cauchy and we may define
\[
T(x) = \lim_n T_n(x).
\]
Then we have
\[
T(x + \lambda y) = \lim_n T_n(x + \lambda y) = \lim_n T_n(x) + \alpha T_n(y) = T(x) + \lambda T(y).
\]
Thus \(T\) is linear. Let us show that
\[
\lim_n \|T - T_n\|_{op} = 0.
\]
Indeed, let \(x \in X\) with \(\|x\| \leq 1\). Then we have
\[
\|T(x) - T_n(x)\| = \|\lim_m T_m(x) - T_n(x)\| \leq \limsup_{m \geq n} \|T_m(x) - T_n(x)\| \\
\leq \sup_{m \geq n} \|T_m - T_n\| \|x\| \leq \sup_{m \geq n} \|T_m - T_n\|.
\]
In particular $\|T\|_{op} \leq \|T - T_1\|_{op} + \|T_1\|_{op}$ is finite and $T$ is continuous. Moreover, $\lim_n d(T, T_n) = 0$ implies that $\lim_n T_n = T$. ■

**Corollary 4.6.** Let $X$ be a normed space. Then $X^* = L(X, \mathbb{R})$ is a Banach space. Moreover, $X^{**} = L(X, \mathbb{R})$ is a Banach space.

**Definition and Remark 4.7.** Let $\iota : X \to X^{**}$ be the linear map given by $\iota(x)(x^*) = x^*(x)$. Then

$$\|\iota(x)\| \leq \|x\|.$$ 

Indeed, the Hahn-Banach theorem (proved in the next course) shows that $\|\iota(x)\| = \|x\|$. A Banach space $X$ is called reflexive if $\iota(X) = X^{**}$, i.e. $\iota$ is surjective. All finite dimensional spaces are reflexive.
5. \( L_p \) spaces

In the following \((\Omega, \Sigma, \mu)\) is a sigma-finite measure space. We define
\[
L_0 = \{ f : \Omega \to \mathbb{R} : \lim_{\alpha \to \infty} \mu(|f| > \alpha) = 0 \}. 
\]
On \( L_0 \) we define the equivalence relation
\[ f \sim g \text{ if } f = g \mu \text{ a.e.} \]
i.e. there exists a set \( F \in \Sigma \) with measure 0 such that \( f(\omega) = g(\omega) \) for all \( \omega \in F^c \).
We define
\[
L_0(\mu) = L_0/\sim
\]

**Proposition 5.1.** (Hw) \( L_0(\mu) \) equipped with the distance
\[
d([f], [g]) = \inf \{ \varepsilon : \mu(|f - g| > \varepsilon) < \varepsilon \}
\]
is a complete metric space.

**Definition 5.2.** \( L_p \) is the set of all measurable functions \( f \) such that \( \int |f|^p \) is finite.

**Lemma 5.3.** (Hölder’s inequality) Let \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( f \in L_p, g \in L_q \). Then \( fg \) is integrable and
\[
|\int fg| \leq \left( \int |f|^p \right)^{\frac{1}{p}} \left( \int |g|^q \right)^{\frac{1}{q}}.
\]

**Proof.** We use the fact that \( g(x) = -\ln(x) \) is convex. Thus for positive numbers \( a, b \) we have
\[
-\ln\left( \frac{a^p}{p} + \frac{b^q}{q} \right) = g\left( \frac{a^p}{p} + \frac{b^q}{q} \right) \leq \frac{1}{p} g(a^p) + \frac{1}{q} g(b^q) = -\ln(a) - \ln(b).
\]
This yields
\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\]

Thus for every \( \omega \in \Omega \) and \( s > 0 \) we find
\[
|f(\omega)||g(\omega)| \leq \frac{|sf(\omega)|^p}{p} + \frac{|s^{-1}g(\omega)|^q}{q}.
\]
Using this for \( s = 1 \), we deduce that \( \int |f|^p < \infty \) and \( \int |g|^q < \infty \) implies \( \int |fg| < \infty \).
Thus we get
\[
|\int fg| \leq \int |fg| \leq \frac{s^p}{p} \int |f|^p + \frac{s^{-q}}{q} \int |g|^q.
\]
We define \( s = \frac{(\int |g|^q)^{1/(p+q)}}{(\int |f|^p)^{1/(p+q)}} \). This implies
\[
s^p \int |f|^p = s^{-q} \int |g|^q.
\]

Hence, we deduce from \( p/(p + q) = 1/q \) and \( q/(p + q) = 1/p \) that
\[
|\int fg| \leq s^p \int |f|^p = (\int |g|^q)^{p/(p+q)}(\int |f|^p)^{1-p/(p+q)}
\]
\[
= (\int |g|^q)^{1/q}(\int |f|^p)^{1/p}.
\]

**Remark 5.4.** Let \( f \) be a measurable function. Then
\[(\int |f|^p)^{1/p} = \sup\{|\int fg| : g \in S(\mu) \int |g|^q \leq 1\}.
\]

**Proof.** Let \( 0 \leq h \leq |f|^p \) be a simple function. This implies \( 0 \leq h^{1/p} \leq |f| \). We write \( h^{1/p} = \sum_i r_i 1_{E_i} \) and define
\[
g(\omega) = (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^\frac{p}{q} \frac{1_{E_i(\omega)}}{|f(\omega)|} f(\omega).
\]

Then, we get
\[
\int fg = (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^\frac{p}{q} \int_{E_i} |f(\omega)|
\]
\[
\geq (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^\frac{p}{q} \int_{E_i} r_i
\]
\[
= (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^\frac{p}{q} \mu(E_i)
\]
\[
= (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^p = (\int h) \frac{1}{\mu}.
\]

On the other hand
\[
\int |g|^q d\mu = (\int |h|)^{-1} \sum_i r_i^p \mu(E_i) = 1.
\]

This yields the assertion.

**Lemma 5.5.** Let \( f \) and \( g \) be measurable functions. Then
\[
(\int |f + g|^p)^{\frac{1}{p}} \leq (\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}}.
\]
Proof. It suffices to consider $1 < p < \infty$. We may assume that the right hand is finite. Let $0 \leq h \leq |f + g|$. Then, we have
\[ h^p = hh^{p-1} \leq (|f| + |g|)h^{p-1}. \]
By Lemma 5.3 we deduce (for $\frac{1}{q} = 1 - \frac{1}{p}$)
\[
\int h^p \leq \int |f|h^{p-1} + \int |g|h^{p-1}
\leq \left( \int |f|^p \right)^{\frac{1}{p}} \left( \int h^{(p-1)q} \right)^{\frac{1}{p}} + \left( \int |g|^p \right)^{\frac{1}{p}} \left( \int h^{(p-1)q} \right)^{\frac{1}{p}}
= \left( \int |f|^p \right)^{\frac{1}{p}} + \left( \int |g|^p \right)^{\frac{1}{p}} \int h^{1 - \frac{1}{p}}.
\]
If $\int h^p = 0$ there is nothing to show. In the other case we obtain
\[
\left( \int h^p \right)^{\frac{1}{p}} \leq \left( \int |f|^p \right)^{\frac{1}{p}} + \left( \int |g|^p \right)^{\frac{1}{p}}.
\]
Taking the sup over all $0 \leq h \leq |f + q|$ we deduce the assertion. \hfill \qed

Theorem 5.6. Let $1 \leq p < \infty$. The space
\[ L_p = \{[f] : f \text{ measurable }, \|[f]\|_p = \left( \int |f|^p \right)^{\frac{1}{p}} \} \]
with the norm $\| \|$ is a Banach space.

Proof. We note first that for $f \sim g$ we have
\[ \int |f|^p = \int |g|^p. \]
Thus $\| \|$ is well-defined. Moreover, we have
\[ \|[\lambda f]\|_p = |\lambda|\|[f]\|_p. \]
By the definition of equivalent classes, we see that
\[ \|[f]\|_p = 0 \iff f = 0 \text{ a.e.} \iff [f] = 0. \]
For $x, y \in L_p$, we pick $f \in x, g \in y$. According to Lemma 5.3, we deduce that $|f + g|^p$ is integrable and hence $x + y = [f + g]$ is in $L_p$ satisfying
\[ \|x + y\|_p = \|[f + g]\|_p = \left( \int |f + g|^p \right)^{\frac{1}{p}} \leq \left( \int |f|^p \right)^{\frac{1}{p}} + \left( \int |g|^p \right)^{\frac{1}{p}} = \|x\|_p + \|y\|_p. \]
Thus $(L_p, \| \|_p)$ is a normed vector space. Let $(x_n)$ be a Cauchy sequence in $L_p$. We may assume
\[ \|x_n - x_{n+1}\|_p \leq 2^{-n\frac{p+1}{p}}. \]
Let $f_n \in x_n$. Then we find
\[ \int |f_n - f_{n+1}|^p \leq 2^{-n(p+1)}. \]

By Chebychev's inequality we deduce
\[ \mu(|f_n - f_{n+1}| > 2^{-n}) 2^{-np} \leq \int |f_n - f_{n+1}|^p \leq 2^{-n(p+1)}. \]

Thus we get
\[ \mu(|f_n - f_{n+1}| > 2^{-n}) \leq 2^{-n}. \]

The convergence Lemma implies that $(f_n)$ is almost everywhere convergent to a measurable function $f$. Define
\[ h = |f_1| + \sum_n |f_{n+1} - f_n|. \]

We want to show that $|h|^p$ is integrable. Indeed, we apply the monotone convergence Lemma and deduce from the triangle inequality in $L_p$
\[ \int |h|^p \leq \liminf_m \int (|f_1| + \sum_{n=1}^m |f_{n+1} - f_n|)^p \leq \liminf_m (\|f_1\|_p + \sum_{n=1}^m \|f_{n+1} - f_n\|_p)^p \leq (\|x_1\|_p + \sum_n \|x_{n+1} - x_n\|_p)^p \leq (\|x_1\|_p + 2)^p < \infty. \]

Moreover, we have $|f_m - f_n|^p \leq |h|^p$ and for $m \geq n$ we get that
\[ (\int |f_m - f_n|^p)^{\frac{1}{p}} \leq \sum_{k=n}^m \|x_{k+1} - x_k\|_p \leq \sum_{k=n}^\infty 2^{-k\frac{n+1}{p}} \leq 2^{1-n}. \]

By the dominated convergence theorem (with majorant $|h|^p$) we deduce that
\[ \int |f - f_n|^p d\mu = \lim_{m \geq n} \int |f_m - f_n|^p \leq 2^{p(1-n)}. \]

Using the triangle and $f_1 \in L_p$, we deduce that $f \in L_p$. Moreover,
\[ \lim_n \|f - x_n\|_p = 0. \]

This completes the proof.

**Proposition 5.7.** Let $1 \leq p < \infty$. The simple functions are dense in $L_p$. For $(\Omega, \Sigma, \mu) = (\mathbb{R}, \mathcal{L}, m)$ the step functions are dense in $L_p$ and the continuous functions are dense in $L_p$. 

Proof. Let $f \geq 0$ such that $\int f^p < \infty$. Let $h_n$ be an increasing sequence of simple functions such that

$$I(f^p - h_n) < 2^{-n}.$$  

Then $h_n$ converges to $f^p$ a.e. and also $h^\frac{1}{p}$ converges to $f$ almost everywhere. Since $|f - h_n|^p \leq |f|^p$, we deduce from the dominated convergence theorem that

$$\lim_n \int |f - h_n|^p = \int 0 = 0.$$  

This proves $\lim_n \| [f] - [h_n] \|_p = 0$. The general assertion follows by considering $f = f^+ - f^-$. For the second assertion, we assume again that $f \geq 0$ and $0 \leq h \leq f$ such that

$$\int |f - h|^p < \varepsilon.$$  

Let $C = \sup \|h\|$. Using a small perturbation, we may also assume that $h$ vanishes in $(-\infty, n] \cup [n, \infty)$. By the applications of Lusin’s theorem, we may find a step function $g$ such that $0 \leq g \leq C$ and

$$\mu(|g - h| > \delta) < \delta.$$  

Then, we get

$$\int_{-\infty}^{\infty} |g - h|^p = \int_{-\infty}^{\infty} 1_{|g - h| > \delta}|g - h|^p + \int_{-\infty}^{\infty} 1_{|g - h| \leq \delta}|g - h|^p$$

$$\leq (2C)^p \mu(|g - h| > \delta) + \delta^p 2n \leq (2C)^p \delta + 2n \delta^p.$$  

Choosing $\delta$ small enough we get $(\int_{-\infty}^{\infty} |g - h|^p)^{\frac{1}{p}} < \varepsilon$. Starting from step functions, continuous functions are achieved as in Lemma 2.8.  

$\blacksquare$