CHAPTER 5

Integration in \( \mathbb{R} \)

1. Absolute continuous functions

A function \( f : [a, b] \to \mathbb{R} \) is called of bounded variation if

\[
\|f\|_{BV} = \sup \left\{ \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| : a = x_0 < x_1 < \cdots < x_n = b \right\}
\]

is finite. We say that \( f \) is absolutely continuous if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for every partition \( a = x_0 < x_1 < \cdots < x_n = b \) and every subset \( J \subset \{1, \ldots, n\} \)

\[
\sum_{i \in J} |x_{i+1} - x_i| < \delta \implies \sum_{i \in J} |f(x_{i+1}) - f(x_i)| < \varepsilon .
\]

**Lemma 1.1.** Let \( f \in L^1[a, b] \) and \( F(t) = \int_a^t f(s) \, dm(s) \). Then \( F \) is of bounded variation and absolutely continuous. Moreover, \( F(a) = 0 \) and \( \|F\|_{BV} \leq \int |f| \).

**Proof.** For a partition \( a = x_0 < x_1 < \cdots < x_n = b \) and \( J \subset \{1, \ldots, n\} \) and \( \varepsilon_i = \frac{F(x_{i+1}) - F(x_i)}{|F(x_{i+1}) - F(x_i)|} \) we have

\[
\sum_{i \in J} |F(x_{i+1}) - F(x_i)| = \left| \int \left( \sum_{i \in J} \varepsilon_i 1_{[x_i, x_{i+1})} \right) f \, dm \right| \leq \int |f| 1_{\bigcup_{i \in J} [x_i, x_{i+1})} \, dm .
\]

Thus for \( J = \{1, \ldots, n\} \) we get

\[
\sum_{i=0}^{n-1} |F(x_{i+1}) - F(x_i)| \leq \int |f| \, dm .
\]

The absolute continuity follows from

\[
\lim_{m(A) \to 0} \int_A |f| \, dm = 0 .
\]

(see exam).
Lemma 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a of bounded variation. Then $f$ is the difference of two monotone functions $f_1, f_2$. If in addition $f$ is absolutely continuous, then $f_1$ and $f_2$ may be assume absolutely continuous.

$$\|f\|_{BV} = f_1(b) + f_2(b) - f(a).$$

Proof. For any partition $\pi$ we define

$$p(f, \pi) = \sum_{i=0}^{n-1} \max\{f(x_{i+1}) - f(x_i), 0\}$$

$$n(f, \pi) = \sum_{i=0}^{n-1} \max\{-f(x_{i+1}) + f(x_i), 0\}$$

$$t(f, \pi) = \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|.$$

Then

$$t(f, \pi) = p(f, \pi) + n(f, \pi)$$

and

$$f(b) - f(a) = \sum_{i=0}^{n} (f(x_{i+1}) - f(x_i)) = p(f, \pi) - n(f, \pi).$$

This implies

$$f(b) - f(a) + n(f, \pi) = p(f, \pi).$$

Now, we take the supremum over all partitions and still have

$$f(b) - f(a) + \sup_{\pi} n(f, \pi) = \sup_{\pi} p(f, \pi).$$

Moreover,

$$\sup_{\pi} t(f, \pi) = \sup_{\pi} [p(f, \pi) + n(f, \pi)] = \sup_{\pi} [2p(f, \pi) - (f(b) - f(a))]$$

$$= \sup_{\pi} p(f, \pi) + \sup_{\pi} p(f, \pi) - (f(b) - f(a)) = \sup_{\pi} p(f, \pi) + \sup_{\pi} n(f, \pi).$$

For $a \leq x \leq b$ we define

$$g(x) = \sup_{\pi=\{a=x_0<...<x_n=x\}} p(f, \pi)$$

and

$$h(x) = \sup_{\pi=\{a=x_0<...<x_n=x\}} n(f, \pi).$$

Then $g$ and $h$ are increasing functions and

$$f(x) - f(a) + h(x) = g(x).$$
This yields
\[ f(x) = g(x) - h(x) + f(a) . \]
Moreover,
\[ \|f\|_{BV} = g(b) - f(a) + h(b) . \]
If in addition \( f \) is absolute continuous then it follows by the definition that
\[ \sum_{i} |x_{i+1} - x_i| < \delta \]
implies
\[ \sup \sum_{i \in J} |g(x_{i+1}) - g(x_i)| \leq \sup_{\sum_{j} |y_{j+1} - y_j| < \delta} \sum_{j} \max \{f(y_{j+1}) - f(y_j), 0\} . \]
Thus we can work with same relation between \( \varepsilon \) and \( \delta \) for \( g \) and \( h \).

**Theorem 1.3.** Let \( F : [a, b] \rightarrow \mathbb{R} \) be an absolute continuous function of bounded variation. Then there exists a function \( f \in L_1[a, b] \) such that
\[ F(t) = F(a) + \int_{a}^{t} f(s)ds \]
and \( \|f\|_1 = \|F\|_{BV} . \)

**Proof.** We may assume \( F(a) = 0 \). Let \( F = F_1 - F_2 \) such that \( F_1 \) and \( F_2 \) are positive
\[ F_1(b) + F_2(b) = \|F\|_{BV} \]
and such that \( F_1, F_2 \) are absolutely continuous. We define the measure on \( A_{\mathbb{R}} \).
\[ \nu((s, t]) = F_1(t) - F(s) . \]
Using the absolute continuity it is not hard to check that
\[ \nu((s, t]) = \sum_{j} \nu((s_j, t_j]) \]
for every disjoint decomposition. Thus \( \nu \) extends to a \( \sigma \) additive measure on the borel sets which is absolutely continuous with respect to the Lebesgue measure. Thus \( \nu \) extends to Lebesgue measurable set. By the Radon-Nikodym theorem we find a measurable function \( f_1 \) such that
\[ F_1((s, t]) = \int_{s}^{t} f_1 dm . \]
Then
\[ \int f_1 dm = F_1(b) - F_1(a) = F_1(b) . \]
We apply the same argument to $F_2$ and find a positive element $f_2$ such that
\[ \int f_2 \, dm = F_2(b) . \]
Thus $f = f_1 - f_2$ satisfies the assertion by Lemma 1.1.