Introduction to real analysis - hw2

Due date: Monday, September 13

i) (30p) Show that
\[ B = \{ x \in \mathbb{R}^2 : d_{SNCF}(x, (0,0)) \leq 1 \} \]
is closed bounded, but not compact (in \((\mathbb{R}^2, d_{SNCF})\)). Let \( u : B \to (\mathbb{R}^m, d_2) \) be an injective map such that
\[ d_2(u(x), u(y)) \leq d_{SNCF}(x, y). \]
Show that there are sequences \((x_n), (y_n) \subset B\) such that
\[ \lim_{n} d_2(u(x_n), u(y_n)) = 0, \]
(Hint: If this is not the case \( \inf d_2(u(x), u(y)) \geq c > 0 \) (why?). However, this leads to a contradiction if you have done your job before correctly).

ii) (20P) Let \((r_n) \subset [0,1]\) be a strictly increasing sequence. Let \((\alpha_n) \subset [0,1]\) be such that \( \sum_n \alpha_n \). We define
\[ f_n(x) = \alpha_n 1_{[r_n,1]}(x) = \alpha_n \begin{cases} 1 & \text{if } x \geq r_n \\ 0 & \text{else} \end{cases} . \]
Show that for every \( x \in [0,1] \)
\[ f(x) = \sum_n f_n(x) \]
is well-defined. Show that \( f \) is upper semi-continuous. Let \( y \in \mathbb{R}^2 \). Show that \( f(x) = -d_{SCNF}(x, y) \) is upper semicontinuous.

iii) (30p) Show that \( C[0,1] \) with \( d_1 \) is not complete with respect to
\[ d_1(f, g) = \int_{0}^{1} |f(t) - g(t)| . \]

iv) (10p) Show that a closed subset of a compact set is compact. (If possible use the definition with open covers).

v) (10P). Show Lindelöf’s theorem in \( \mathbb{R}^m \): Every open subset of \( \mathbb{R}^m \) is a countable union of open balls.
Introduction to real analysis - hw1

Due date: Monday, September 13

i) Show that
\[ B = \{ x \in \mathbb{R}^2 : d_{SNCF}(x, (0,0)) \leq 1 \} \]
is closed bounded, but not compact (in \((\mathbb{R}^2, d_{SNCF})\)). Let \( u : B \to (\mathbb{R}^m, d_2) \) such that
\[ d_2(u(x), u(y)) \leq d_{SNCF}(x, y) . \]
Show that there exist sequences \((x_n), (y_n) \subset B\) such that
\[ \lim_n \frac{d_2(u(x_n), u(y_n))}{d_{SNCF}(x_n, y_n)} = 0 . \]

ii) Let \((r_n) \subset [0,1]\) be a strictly increasing sequence. Let \((\alpha_n) \subset [0,1]\) be such that \(\sum_n \alpha_n\). We define
\[ f_n(x) = \alpha_n 1_{[r_n,1]}(x) = \alpha_n \begin{cases} \alpha_n 1 & \text{if } x \geq r_n \\ 0 & \text{else} \end{cases} . \]
Show that for every \(x \in [0,1]\)
\[ f(x) = \sum_n f_n(x) \]
is well-defined. Show that \(f\) is upper semi-continuous.

(1) Show that \(C[0,1]\) with is not complete with respect to
\[ d_1(f, g) = \int_0^1 |f(t) - g(t)| . \]
Due date: Wednesday, September 20

(1) (20P) Show that $[0, 1]$ is compact by verifying the definition.

(2) i) (10P) Given an example of a continuous function and a closed set $C$ such that $f(C)$ is not closed.
   ii) (10P) Show that image of a compact set under a continuous map is compact.
   iii) (15P) Show that image of a relatively compact set under a continuous map is relatively compact.
   iv) (10P) Let $X$ be a compact metric space and $f : X \to Y$ be continuous and bijective. Show that the inverse function $f^{-1}$ is continuous.

(3) (30P) Let us consider the set

$$Lip_c = \{ f \in C[0, 1] : |f(x) - f(y)| \leq c|x - y| \}.$$ 

Show that $Lip_c$ is not relatively compact but

$$F = Lip_c \cap B(0, 1) = \{ f \in Lip_c : \sup_{0 \leq x \leq 1} |f(x)| \leq 1 \}$$

is relatively compact in $C[0, 1]$. As an application, we consider $I : C[0, 1] \to C[0, 1]$ given by

$$I(f)(t) = \int_0^t f(s) ds.$$ 

Show that

$$\{ I(f) : \sup_{0 \leq x \leq 1} |f(x)| \leq 1 \}$$

is relatively compact in $C[0, 1]$.

(4) (20P) No 36a) on page=161: A point $x$ in a metric space is called isolated if the set $\{ x \}$ is open. Show that a complete metric space without isolated points has an uncountable number of points.
Due date: Monday, September 27

(1) (10P)
   (a) Let $F$ is nowhere dense if and only if every nonempty open set $O \cap F^c$ contains an open ball.
   (b) Show that the countable union of meager sets is meager.

(2) (25P) We want to show that for an upper semicontinuous function $f : X \to \mathbb{R}$ on a complete metric space the set of continuity points is dense.
   (a) Let $a < b$ in $\mathbb{R}$. Show that
   \[
   A_{a,b} = \{ x \in X : f(x) \geq b \text{ and } \exists_{(x_j), \lim_j x_j = x} \lim_j f(x_j) \leq a \}
   \]
   is closed.
   (b) Show that the sets $A_{[a,b]}$ have nonempty interior.
   (c) For $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ we define $B_{k,m} = A_{\left(\frac{k-1}{m}, \frac{k}{m}\right]}$. Show that for $x \in \bigcap_{k,m} B_{k,m} f$ is continuous at $x$. Show that for complete $X$ the set of continuity points is dense.

(3) (20P) Let $f : X \to Y$ be a function. Define
   \[
   \omega_\delta(f)(x) = \sup \{ d'(f(y), f(z)) : d(x, y) < \delta \text{ and } d(x, z) < \delta \}
   \]
   Let $\varepsilon > 0$. Show that $\{ x : \omega_\delta(f)(x) < \varepsilon \}$ is open. Conclude that the set $C(f)$ of continuity points can be written as a countable intersection of open sets.

(4) (25) Let $Y$ be a separable metric space and $X$ a complete metric space. Let $f : X \to Y$ be a function such that $f^{-1}(\overline{B}(y, \varepsilon))$ is closed for every $y \in Y$ and $\varepsilon > 0$. Then the sets of continuity points is dense in $X$.

(5) (20P) Let $X$ be a set with an uncountably many elements (you might work with $\mathbb{R}$). Show that
   \[
   A = \{ S \subset X : S \text{ or } |X \setminus S| \text{ is countable} \}
   \]
   is a $\sigma$-algebra.
Due date: Monday, October 4

(1) (15P) Show that for set $E \subset \mathbb{R}$ with $m^*(E) < \infty$ and $\varepsilon > 0$ there exists a compact set $C \subset E$ such that $m(C) \leq m(E) < m(C) + \varepsilon$. Conclude that for measurable $E$ we have $m(E \setminus C) < \varepsilon$.

(2) (20P) Let $\mu$ be a $\sigma$-additive measure on $B$ and $(E_j)$ events in $B$.
   (a) Show that if $E_1 \supset E_2 \cdots$ and $m(E_1) < \infty$, then
   $$\mu\left(\bigcap_j E_j\right) = \lim_j \mu(E_j).$$
   Show that the assumption $\mu(E_1) < \infty$ is really needed.
   (b) Show that if $E_1 \subset E_2 \cdots$, then
   $$\mu\left(\bigcup_j E_j\right) = \lim_j \mu(E_j).$$

(3) (30P) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function such that $f(x) = \lim_{y \to x^-} f(y)$. (That means $f$ is continuous from the left.) We define
   $$m_f([a, b)) = f(b) - f(a).$$
   Show that for every interval $I = [a, b]$ we have
   $$f(b) - f(a) \leq \inf\left\{ \sum_j m_f(I_j) : I \subset \bigcup_j I_j \right\}.$$ 
   Here the infimum is taken over right open intervals $I_j = [a_j, b_j)$ (in principle we allow $a_j = -\infty$ and consider $(-\infty, b_j)$ as half open). Show that
   $$f(b) - f(a) = \inf\left\{ \sum_j m_f(I_j) : [a, b) \subset \bigcup_j I_j \right\}.$$ 

(4) (30P) We will need an estimate using Sterling’s formula, namely
   $$\lim_n 2^{-2n} \left(\frac{2n}{n}\right) = 0.$$ 
   (You can amuse yourself in finding Sterling’s formula and how to deduce that.)
   (a) Let $X_n = \{-1, 1\}^n$ and $\mu_n(A) = 2^{-n}|A|$ (where $|A|$ is the cardinality of $A$). Show that
   $$\mu_n\left(\{\varepsilon_1, \ldots, \varepsilon_n \} : \sum_{i=1}^n \varepsilon_i = k\right) = 2^{-n} \left(\frac{n}{n+k}\right).$$
Deduce from this that for even $n$

$$
\mu_n(\{(\varepsilon_1, \ldots, \varepsilon_n) : \sum_{i=1}^{n} |\varepsilon_i| \leq k\}) = 2k2^{-n}\left(\frac{n}{2}\right).
$$

(b) On $X_\infty = \{-1, 1\}_\infty$ we denote by $\mu$ the extension of then $\mu_n$’s (explained in class). Show that for every $k$

$$
\mu(\{(\varepsilon_1, \ldots) : \sup_n \left|\sum_{j=1}^{n} \varepsilon_j\right| \leq k\}) = 0.
$$

(c) Let $X_\infty$. Show that

$$
\mu(\{(\varepsilon_1, \ldots) : \lim_n \sum_{j=1}^{n} \varepsilon_j \text{ exists } \}) = 0.
$$
(5) Real Analysis-Homework 6

**Due date:** Monday, October 18

(1) Let us denote by \( p_k(t) = t^k, \ k \in \mathbb{N}_0 \) the polynomials. Show that
\[
\{a_0p_0 + a_1p_1 + a_2p_2 : a_0, a_1, a_2 \in \mathbb{R}\}
\]
is a closed subset of \( C[0,1] \). (Hint: Show that for \( R > 0 \) the set
\[
\{(a_0, a_1, a_2) : \sup_{0 \leq t \leq 1} |a_0 + ta_1 + t^2a_2| \leq R\}
\]
is closed and bounded.)

(2) Let \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[
\phi(x_1, x_2) = (2x_1, \frac{1}{2}x_2).
\]
Show that for every Lebesgue measurable subset \( E \) of \( \mathbb{R}^2 \),
\[
m(\phi^{-1}(E)) = m(E).
\]

(3) (a) Use an enumeration of the rational numbers in \([0,1]\) and construct an
open set of measure \(< \frac{1}{n}\) which contains the rational numbers.
(b) Construct a meager subset of \([0,1]\) of measure 1.
Due date: Monday, October 25

(1) (25P) Let $P \subset [0,1]$ be the non measurable set constructed in class (as set of representatives of $x \sim y$ iff $x - y \in \mathbb{Q}$). Show that for every measurable subset $E \subset P$ we have $m(E) = 0$. (Royden: 3.15)

(2) (25P) Show that every set with $0 < m^*(A) < \infty$ contains a non measurable set (Royden 3.16).

(3) (25P) Show Proposition 24 on page 73 (Royden): Let $E$ be a measurable set of finite measure and $(f_n)$ a sequence of measurable functions which converges to a real valued function $f$ a.e. Show that for every $\varepsilon > 0$ and $\delta > 0$, there exists a subset $A \subset E$ such that $m(A) < \delta$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and $x \notin A$

$$|f_n(x) - f(x)| < \varepsilon.$$

(4) (25P) (Royden 3.30) Prove Egoroff’s theorem: Let $E$ be a measurable set of finite measure and $(f_n)$ a sequence of measurable functions which converges to a real valued function $f$ a.e. Show that on a set of large measure $(f_n)$ converges uniformly to $f$. 

Real Analysis-Homework 8

Due date: Monday, November 1

(1) (15P) Let $\Omega = \mathbb{R}$, $\Sigma = \{A : A is countable or $A^c$ is countable\}$ and

$$
\mu(A) = \begin{cases} 
\infty & A^c \text{ is countable} \\
0 & A \text{ is countable} 
\end{cases}.
$$

Show that $\mu$ is $\sigma$-additive. Consider $f = 1_\mathbb{R}$. Show that

$$
I(f) = 0 \quad \mu(\{x \in \mathbb{R} : f(x) \geq \frac{1}{2}\}) = \infty.
$$

(2) Let $(\Omega, \Sigma, \mu)$ be a measure space (not necessarily $\sigma$-finite). We will now say that a function $f : \Omega \to [-\infty, \infty]$ is measurable in the strong sense if there exists a set $F \in \Sigma$ of measure 0 and a sequence $(g_n)$ of simple functions such that

$$
f(\omega) = \lim_{n} g_n(\omega)
$$

holds for all $\omega \in F^c$.

(a) (15P) Let $f \geq 0$ and measurable in the strong sense. Show that for every $\lambda > 0$ the set $E_\lambda = \{\omega \in \Omega : f(\omega) \geq \lambda\}$ is $\sigma$-finite, i.e. there exists $G_n \in \Sigma$ with finite measure such that $E_\lambda = \bigcup_n G_n$. (Hint: use the functions $h_n = \inf_m g_n$.

(b) (10P) Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Show that every measurable function is measurable in the strong sense. (Hint: use the functions $f_\varepsilon$ below).

(3) (20P) Let $(\Omega, \Sigma, \mu)$ be a measure space such that $\mu(\Omega) < \infty$. Let $f \geq 0$ and $\varepsilon > 0$. Consider

$$
f_\varepsilon = \sum_{k=0}^{\infty} (k\varepsilon) 1_{\{k\varepsilon \leq f < (k+1)\varepsilon\}}.
$$

Show that

$$
I(f) = \lim_{\varepsilon \to 0} I(f_\varepsilon).
$$

Conclude that

$$
I(f) = \inf \left\{ \sum_k r_k \mu(E_k) : f \leq \sum_k r_k 1_{E_k} \right\}.
$$

(4) In this exercise we want to establish the link between areas and integrals.

Let $(\Omega, \Sigma, \mu)$ be a finite measure space ($\mu(\Omega) < \infty$). On $\tilde{\Omega} = [0, \infty) \times \Omega$ we consider the algebra $A$ generated by the sets $[s, t) \times E$, $0 \leq s \leq t \leq \infty$, $E \in \Sigma$. 

(a) (10P) Show that every element $F$ in $A$ can be written as

$$F = \bigcup_{k=1}^{m} G_k \times E_k$$

such that the $E_k$'s are disjoint and $G_k \in \mathbb{A}_R$. We then define

$$\nu(F) = \sum_{k} m(G_k) \mu(E_k).$$

It can be shown that for every $F \in A$ and for every disjoint union $F = \bigcup_{j} F_{j}$ of disjoint sets in $A$ we have

$$\nu(F) = \sum_{j} \nu(F_{j}).$$

Therefore we may extend $\nu$ to a $\sigma$-additive measure on the $\sigma$-algebras $\tilde{\Sigma}$ of measurable subsets (in the Caratheodory sense) of $[0, \infty] \times \Omega$.

(b) (5P) Let $E \in \Sigma$ be a set of measure 0. Show that

$$\nu([0, \infty) \times E) = 0.$$  

(c) (15P) Let $f : \Omega \to [0, \infty]$ be measurable such that $I(f) < \infty$. Show that the graph of $f$

$$G(f) = \{(r, \omega) : 0 \leq r \leq f(\omega)\}$$

has finite measure with respect to $\nu$ and satisfies $\nu(G(f)) \leq I(f)$.

(Hint for a simple function $h$ we have $\nu(G(h)) = I(h)$.

(d) (10P) Let $f : \Omega \to [0, \infty]$ be measurable and assume

$$\nu(G(f)) < \infty.$$  

Show that $I(f) \leq \nu(G(f))$. (Hint: consider a simple function $0 \leq h \leq f$.)
Real Analysis-Homework 9

Due date: Monday, November 15

(1) (a) (10P) Let $(x_k)$ be a sequence of real numbers such that

$$f((\alpha_k)) = \sum_{k=1}^{n} x_k \alpha_k$$

exists for all $(\alpha_k) \subset \ell_2$. Show that $\sum_k |x_k|^2$ is finite.

(b) (20P) Let $X$ be a Banach space and $(f_n) \subset X^*$ be a sequence of linear functionals such that

$$f(x) = \lim_{n} f_n(x)$$

converges for all $x \in X$. Show that $f$ is continuous.

(2) (30P) Problem 18 in chapter 4 p=94 in Royden.

(3) (30P) Problem 19 in chapter 4 p=94 in Royden.
Real Analysis-Homework 10

Due date: Wednesday, December 1

(1) (a) Let \( x_1, ..., x_n \in \mathbb{R} \). Show that
\[
\sup_i |x_i| \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{p}} \sup_{i=1}^{n} |x_i|.
\]
(b) Let \( f = \sum_i x_i 1_{E_i} \) be a simple function. Show that
\[
\lim_{p} \|f\|_p = \|f\|_{\infty}.
\]
(c) Let \( f \) be measurable function such that \([f] \in L_\infty\) show that
\[
\lim_{p} \|f\|_p = \|f\|_{\infty}.
\]
Hint: For a finite measure space you may use suitable simple function to approximate \( f \) from below and above. The general \( \sigma \)-finite case requires an additional approximation argument.

(2) (a) Let \( V = \{(x_n) : \exists k \in \mathbb{N} \forall n > k x_n = 0\} \) the space of finite sequences. We use
\[
\|(x_n)\|_{\infty} = \sup_n |x_n|.
\]
on \( V \). Show that
\[
\phi((x_n)) = \sum_n x_n
\]
is a linear map on \( V \) which is not continuous.
(b) Let \((V, \|\|)\) be a normed space and \( \phi : V \to \mathbb{R} \) be a linear map. Show that \( \phi \) is continuous if and only if
\[
\{x \in V : \phi(x) = 0\}
\]
is closed. (Hint: One implication is easy. For the other implication you may assume that there is a \( x_0 \in V \) with \( f(x_0) = 1 \). Then \( d = \inf \{\|x_0 - y\| : \phi(y) = 0\} > 0 \) (why?). Use this to show that for arbitrary \( x \) with \( f(x) = 1 \) we have \( \|x\| \geq d \) because \( \phi(x - x_0) = 0 \). Conclude).
Real Analysis-Homework 11

**Due date:** Monday, December 6

(1) Royden 5.12 (p=110)
(2) Royden 5.14 (p=111)
(3) Royden 5.15 (p=111)
(4) Royden 5.17 (p=111) (Assume in addition that \( g \) is monotone increasing).