Exam 1

Name:

1. What is the probability of a full house, explain!
   
   **Solution:**  \[ \frac{\binom{13}{1} \binom{4}{1} \binom{12}{1} \binom{4}{2}}{\binom{52}{5}}. \]
   
   We first select the kind for the triple and then the kind for the pair!

2. What is the probability to avoid at least one suits in bridge? Explain!
   
   **Solution:** Let \( F_j \) be the event of avoiding suits \( j \), \( j = 1, \ldots, 4 \). Then
   \[
P(E) = 4P(F_1) - 6P(F_1F_2) + 3P(F_1F_2F_3) - P(F_1F_2F_3F_4) = \frac{4\binom{39}{13} - 6\binom{26}{13} + 3}{\binom{52}{5}}.
   \]

3. Let \((\Omega, \Sigma, P)\) be a probability space and \( O_1, O_2, O_3 \) be three disjoint events such that \( O_1 \cup O_2 \cup O_3 \). Let \( H \) be an event. Show that
   \[
P(H) = \sum_{j=1}^{3} P(H|O_j)P(O_j).
   \]
   
   **Solution:** By the axioms of probability we have
   \[
P(H) = P(H \cap O_1) + P(H \cap O_2) + P(H \cap O_3).
   \]
   Now \( P(H \cap O_j) = P(H|O_j)P(O_j) \) by the definition of the conditional probability.

4. We have a medical test has 60% efficiency and 3% false positives. 10% of the tested population have the disease. What is the probability of getting tested with a negative result provided the patient has the disease.
   
   **Solution:**
   \[
P(D|\neg) = \frac{P(\neg \cap D)}{P(\neg)} = \frac{P(\neg|D)P(D)}{P(\neg|D)P(D) + P(\neg|D^c)P(D^c)}
   \]
   \[
   = \frac{0.03 \times 0.1}{0.03 \times 0.1 + 0.40 \times 0.9} = \frac{0.03 \times 0.1 + 0.40 \times 0.9}{0.03 \times 0.1 + 0.40 \times 0.9} = \frac{1 + 40 \times 0.9}{121} = \frac{1}{121}.
   \]
5. Let $E$ and $F$ be independent and $P(E) = p$, $P(EF^c) = p/3$. What is $P(F)$?

**Solution:** $p/3 = P(EF^c) = P(E)P(F^c) = pP(F^c)$. Thus $P(F^c) = 2/3$ and $P(F) = 1/3$.

6. Let $Q_n$ denote the probability that no run of consecutive 3 heads appears in $n$ tosses of a fair coin. Show that

$$Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3} + \frac{1}{8}$$

and $Q_0 = Q_1 = Q_2 = 1$. Calculate $Q_4$! (Hint: condition on the first success, then on the second, then on the third.).

**Solution:** Clearly $Q_0 = Q_1 = Q_2 = 1$ because we can no have 3 success in less than 3 plays. Let $T_1, T_2, T_3$ the outcome of the first coin. Let $E_n$ the event of 3 consecutive successes Then we have

$$Q_n = P(E_n|T_1 = T)\frac{1}{2} + P(E_n|T_1 = H)\frac{1}{2}$$

$$= Q_{n-1}\frac{1}{2} + P(E_n|T_2 = T, T_1 = T)\frac{1}{4} + P(E_n|T_2 = H, T_1 = H)\frac{1}{4}$$

$$= Q_{n-1}\frac{1}{2} + Q_{n-2}\frac{1}{4}$$

$$+ P(E_n|T_3 = T, T_2 = H, T_1 = T)\frac{1}{8} + P(E_n|T_3 = H, T_2 = H, T_1 = H)\frac{1}{8}$$

$$= Q_{n-1}\frac{1}{2} + Q_{n-2}\frac{1}{4} + Q_{n-3}\frac{1}{8} + \frac{1}{8}.$$

Then $Q_3 = 1/8$ and $Q_4 = 1/8 + 1/16$. 

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Exam 2-Math 461

Name:                      Good luck          May 3, 2015

1a | 1b | 2a | 2b | 3a | 3b | 3c | 4a | 4c | total
---|---|---|---|---|---|---|---|---|-----
10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10  | 100  

1. (a) Let $X$ be a geometric random variable with parameter $p$. Calculate $EX$ and $V(X)$.

(b) Let $Y$ be negative binomial random variable with parameter $(r, p)$. Calculate $EY$ and $V(Y)$.

We have $P(X = k) = (1 - p)^{k-1}p$ and hence

$$\sum kP(X = k) = p \sum k(1 - p)^{k-1} = pF'(q)$$

where $q = 1 - p$, $F(q) = 1/(1 - q) = \sum k q^k$. Thus $F'(q) = 1/(1 - q)^2 = 1/p$. Thus

$$EX = p/p^2 = 1/p$$

Similarly,

$$EX^2 = \sum k^2(1-p)^{k-1}p = p(1-p)F''(q)+EX = 2p(1-p)/p^3+1/p = 2(1-p)/p^2+1/p$$

Thus

$$V(X) = 2(1 - p)/p^2 + 1/p - 1/p^2 = 1/p^2 - 1/p = (1 - p)/p^2.$$ 

For b) we just recall that $Y = \sum_{j=1}^r X_j$ is given by a sum of independent copies and hence

$$EY = r/pV(Y) = r(1 - p)/p^2.$$ 

2. (a) Let $E$ be an event with $P(E) = p$ and $X = 1_E$. Calculate $F_X(t)$ for all $t \in \mathbb{R}$. 

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(b) Let $E_n$ with $\lim_n P(E_n) = P(E)$. Show that $X_n = 1_{E_n}$ converges to $X = 1_E$ in distribution.

We see that $P(X > t) = \begin{cases} 
0 & t \geq 1 \\
(1 - p) & 1 > t \geq 0 \\
1 & t < 0
\end{cases}$ Thus $F_X(t) = 0$ for $t < 0,$

$F_X(t) = (1 - p), 0 \leq t < 1, F_X(t) = 1$ for $t > 1$. In part b) we just write $p_n = P(E_n)$ and $1 - p_n = 1 - P(E_n)$. Clearly, $\lim_n 1 - p(E_n) = 1 - p$, and hence we have convergence in distribution.

3. The life time of a computer follows a continuous distribution with density $Ce^{-\lambda x}$ for $x \geq 0$ (and 0 for $x < 0$).

(a) Find $C$.

(b) Find a function $f_X$ on $[0, 1]$ such that

$$f_X(U) = X$$

holds in distribution for the uniform random variable $U$ on $[0, 1]$. (Hint: The term $-\ln(1 - s)$ comes up.)

(c) Calculate $EX$ in two different ways.

Indeed, $y = \lambda x$, $dy = \lambda dx$ gives

$$\int_0^\infty e^{-\lambda x} \, dx = \int_0^\infty e^{-y} \frac{dy}{\lambda} = 1/\lambda .$$

Thus $C = \lambda$. For b) we have to find

$$P(X > t) = \int_t^\infty e^{-\lambda x} \, dx = \int_\lambda^\infty e^{-y} \, dy = e^{-\lambda t} ,$$

the inverse function of

$$F_X(t) = \begin{cases} 
0 & t \leq 0 \\
1 - e^{-\lambda t} & t > 0
\end{cases} .$$

Hence $s = 1 - e^{-\lambda t}$ means

$$-\lambda t = \ln(1 - s)$$

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or \( t = -\ln(1 - s)/\lambda \). Then \( f_X(s) = -\ln(1 - s)/\lambda \) does the job.

c) We have \( EX = \int x e^{-\lambda x} \lambda dx = 1/\lambda \int_{0}^{\infty} ye^{-y}dy = 1/\lambda \). Alternatively

\[
EX = \int s \frac{-\ln(1 - s)}{\lambda} ds = 1/\lambda
\]

and \( EX = \int P(X > t) dt = \int e^{-\lambda t} dt = 1/\lambda \).

4. Let \((E_i)_{i=1}^{n}\) be independent sets and \(X_i = 31_{E_i} + 21_{E_i^c}\). Let \(X = \sum_{i=1}^{n} X_i\).

(a) Calculate \(EX\) and \(V(X)\).

(b) Use the Poisson-paradigm to show that

\[
P(X = k) \approx P(Y = k)
\]

where \(Y = 2n + Y_\lambda\), \(Y_\lambda\) is Poisson with parameter \(\lambda = \sum_i P(E_i)\). Hint: Rewrite \(X_i - 2\).

We have \(X_i = 2 + 1_{E_i}\). \(EX_i = 2 + P(E_i)\) and hence \(EX = 2n + \sum_i P(E_i)\).

Also

\[
V(X_i) = E(X_i - EX_i)^2 = E(1_{E_i} - P(E_i))^2 = P(E_i) - P(E_i)^2 = P(E_i)(1 - P(E_i))
\]

By independence \(V(X) = \sum_i P(E_i)(1 - P(E_i))\). Finally the Poisson paradigm applies to the (strongly) independent sets \(E_i\) and hence

\[
P(X = k) = P(X = 2n = k - 2n) \approx P(Y_\lambda = k - 2n) = P(Y = k).
\]