1. Conditional probability and Independence

The starting point in Unit 3 is the notion ‘pull of data set’. Let us consider a discrete random variable $X$ with frequencies $1/3$ for 1, $1/4$ for 1.1, $1/4$ for 1.2 and $1/6$ for 5. The probability for the data set \{1, 1.1, 1.2\} is the probability that one of these events occur. In our case this $1/3 + 1/4 + 1/6 = (4 + 3 + 2)/12 = 3/4$.

In general for a set $A$ of outcomes

$$\sum_{o \in A} \nu_k = P(X \in A).$$

CMS uses the notation $P[A, X]$.

For example if we take the random variable $X(x) = \sin(2\pi x)$ on $[0, 1]$, then

$$\text{Prob}(X > \frac{1}{\sqrt{2}})$$
corresponds to a angles of 90 degree out of 360 and hence occurs with probability $1/4$.

**Proposition 1.1.** The mapping $A \mapsto P(X \in A)$ satisfies the properties given by a probability space. In other words the function $\mu(A) = P(X \in A)$ defines a probability measure on the set of events.

One of the most puzzling concepts in probability is ‘conditional probability’

**Definition 1.2.** Let $X$ be a random variable and $A, B$ are sets. Then the probability of $A$ given $B$ with respect to $A$ is defined as

$$P(A \in X|B) = \frac{P(X \in A \cap B)}{P(X \in B)}.$$

In full generality

$$\mu(A|B) = \frac{\mu(A \cap B)}{\mu(B)}.$$

The main mathematical trick here is

$$P(A \cap B) = P(A|B)P(B).$$
1.1. **Monty Hall.** The main confusing example in daily live is the game show problem:

**Problem 1.3.** A candidate has three doors to choose from. Behind one door is car. The showmaster reveals an empty door. Should the candidate switch?

**Simple solution:** Your probability of choosing the a car-lees door is $1/3$. If you always switch that, that is the probability of not getting a car. Thus switching leads to a $2/3$ probability of winning a car.

For a more elaborate solution we need

**Theorem 1.4** (Bayes Law).

$$P(C|D) = \frac{P(D|C)P(C)}{P(D)}.$$ 

**Proof.** Indeed,

$$P(C|D)P(D) = P(C \cap D) = P(D|C)P(C).$$

The next point is careful model of the possible events. This are given by pairs $(a, b)$, where $a$ is the door chosen and b is door revealed. The initial probabilities are $1/3$ for $a = d_1, d_2$ or $d_3$. We will assume that the candidate chooses door $d_1$, and we know that the showmaster has to choose a different door and without a car.

1. $(d_1, d_2)$ and $(d_1, d_3)$ have probability $1/6$. Indeed, with probability $1/3$ the car is behind $d_1$ and then with equal probability the showmaster reveals $d_2$ or $d_3$.
2. $(d_2, d_3)$ has probability $1/3$. If the car is behind $d_2$ and the candidate chooses $d_1$, the showmaster has to reveal $d_3$.
3. $(d_3, d_2)$ has probability $1/3$. If the car is behind $d_3$ and the candidate chooses $d_1$, the showmaster has to reveal $d_2$.

Let $C$ be the event that the car is behind $d_2$ and $D$ be the event that Monty open door $d_3$. Then

$$P(D|C) = 1$$

and

$$P(D) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}.$$ 

Hence

$$P(C|D) = \frac{P(D|C)P(C)}{P(D)} = \frac{1/3 \times 1}{1/2} = \frac{2}{3}.$$
With 2/3 probability changing the door wins the car.

2. Independence

Events $A, B$ are independent if

$$P(A)P(B) = P(A \cap B)$$

or equivalently

$$P(A|B) = P(A).$$

Random variables $X$ and $Y$ are independent if

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

for all $A, B$.

**Remark 2.1.** What is the connection to G.2.i (3.4 try2)?

**Theorem 2.2.** For independent random variables

$$E_{XY} = E_{XEY}$$

and

$$Var(X + Y) = Var(X)Var(Y).$$

**Proof.** Let us assume that $X$ and $Y$ are discrete with frequencies

$$\nu(X = v_k) = a_k \quad \nu(Y = w_j) = b_j.$$

Then

$$Prob(X = v_k, Y = w_j) = a_kb_j.$$

We get

$$E_{XY} = \sum_{k,j} v_kw_jP(X = v_k, Y = w_j) = \sum_{k,j} v_kw_jP(X = v_k)P(Y = w_j)$$

$$= (\sum_k v_kP(X = v_k))(\sum_j w_jP(Y = w_j)).$$

For general $X$ and $Y$, we use a mathematical trick. One can find discrete random variable $\tilde{X}, \tilde{Y}$ such that $X - \tilde{X}$ is small and $Y - \tilde{Y}$ is small and $\tilde{X}$ and $\tilde{Y}$ are still independent. Then one applies the previous calculation for $\tilde{X}$ and $\tilde{Y}$. The second formula now follows from

$$Var(X + Y)^2 = V(X)^2 + V(Y)^2 + 2(EXY - E_{XEY})$$

which we observed earlier. ■
Problem 2.3. Find two random variables which violate the formula for the variance.

2.1. Practical examples.

Problem 2.4. Roll the dice three times. What is the probability of finding even numbers all the time?

Solution: For a single shot \( P(X = \text{even}) = 1/2 \). If we assume that luck starts fresh for every rolling of the dice we get \( P(X_1 = \text{even}, X_2 = \text{even}, X_3 = \text{even}) = 1/8 \).

Problem 2.5. Roll the dice three times. What is the probability to get a strictly increasing or decreasing pattern?

Solution: We only do increasing. Let start with two rolls of a dice with \( m \) edges. We have \( m^2 \) pairs and \( m \) of them are with equal numbers. Thus we get \( a_m = \frac{m^2 - m}{2} \) possibilities to get an increasing number. Now for three options we get
\[
\sum_{j=1}^{4} a_{6-j+1} = \frac{25 - 5}{2} + \frac{16 - 4}{2} + \frac{9 - 3}{2} + \frac{4 - 2}{2} = 10 + 6 + 3 + 1 = 20.
\]
Thus the probability is \( \frac{20}{216} = \frac{5}{54} \). For strictly decreasing the result is the same, and the intersection is 0.

Problem 2.6. How can we do this for large number rolls?

Solution: We write
\[
6^n = (1 + 1 + \cdots + 1)^n = \sum_{j_1,\ldots,j_n=1}^{6} = \sum_{j_1,\ldots,j_n\text{ all different}} + \text{other terms}.
\]
Then we compare this with the binomial theorem
\[
(x_1 + \cdots + x_6)^n = \sum_{l_1+\cdots+l_6=n} \binom{n}{l_1,\ldots,l_6} x_1^{l_1} \cdots x_6^{l_6}.
\]
Also we have
\[
(x_1 + \cdots + x_6)^n = \sum_{j_1,\ldots,j_n} x_1^{j_1} \cdots x_6^{j_6}.
\]
Hence the solution is
\[
\frac{1}{n!} \sum_{l_1+\cdots+l_6=n, l_r > 0} \binom{n}{l_1,\ldots,l_n}.
\]
Maybe this can be improved, but that we do later in the semester. ■
Remark 2.7. More concrete problems 3.5.


Problem 2.9. Let a dice roll two times with outcome $X_1, X_2, X_3$. Are

$$X = X_1 + X_2 \quad \text{and} \quad Y = X_2 + X_3$$

independent? What about $X_1$ and $Y$?

Remark 2.10. Problems discussed in ‘Try’

(1) Try1 $\Prob(X \in A)$
(2) Try2 Indicator function
(3) Try3 Practical Independence for wired networks
(4) Try4 Conditional probability with dice
(5) Try5 Tatoos and tongue rings
(6) Try6 False positives
(7) Try7 Independence, again
(8) Try8 Cards and conditional probability
(9) Try9 Playing craps (preparing for law of large numbers)
(10) Try10 Gambling problems
(11) Try11 Birthday problem
(12) Try12 Strange mind of professor swift
(13) Try13 Actual science problems