1. practice problem for take home

(1) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be Lipschitz, i.e.
\[
|F(x, y) - F(x', y')| \leq L(|x - x'|^2 + |y - y'|^2)^{1/2}.
\]
For a fixed $g$ Consider the map $\Phi : C([0, \infty)) \to C([0, \infty))$
\[
\Phi(f)(x) = g(x) + \int_0^x F(t, f(t))dt.
\]

Solution: Using a standard estimate on the integral we have
\[
|\Phi(f_1)(x) - \Phi(f_2)(x)| \leq \int_0^x L|f_1(t) - f_2(t)|dt
\]
\[
\leq L \int_0^x e^{at}dt d_a(f_1, f_2)
\]
\[
\leq L e^{ax} - 1 \leq \frac{L}{a} e^{ax}.
\]
Thus we have a) and b)

Let us show that $C([0, \infty))$ with this metric is complete. Let $f_n$ be
a Cauchy sequence such that $d_a(f_n, f_m) \leq \varepsilon$ holds for $n, m \geq n(\varepsilon)$. For
every finite $x_0$ we note that
\[
\sup_{0 \leq x \leq x_0} |f(x) - g(x)| \leq e^{ax_0} d_a(f, g).
\]
This $(f_n)$ is Cauchy in $C[0, x_0]$ and hence
\[
f(x) = \lim_n f_n(x)
\]
is continuous in $[0, x_0]$, and in fact on $[0, \infty)$. Note also that
\[
|f_n(x) - f(x)| e^{-ax} = \lim_m |f_n(x) - f_m(x)| e^{-ax} \leq \varepsilon
\]
for $n \geq n(\varepsilon)$. Thus indeed, $f \in C[0, \infty)$. Finally, we have
\[
f(x) = \Phi(f)(x) = g(x) + \int_0^x F(t, f(t))dt
\]
and hence $f$ is differentiable provided $g$ is, and
\[
f'(x) = g'(x) + F(x, f(x))
\]
solves a beautiful ODE.
(a) Show that
\[
|\Phi(f_1)(x) - \Phi(f_2)(x)| \leq \frac{L}{a} (e^{ax} - 1) \sup_t |f_1(t) - f_2(t)| e^{-at}.
\]
(b) On $C([0, \infty))$ we use the distance

$$d_a(f_1, f_2) = \sup_x |f_1(x) - f_2(x)| e^{-ax}.$$ 

Show that for $a \geq 2L$

$$d_a(\Phi(f_1), \Phi(f_2)) \leq \frac{1}{2} d_a(f_1, f_2).$$

(c) Assume you could apply the contraction principle (or Banach Fixpoint Theorem) and obtain a fixpoint $\Phi(f) = f$. Find the differential equation satisfied by $f$.

(d) What is missing before you can apply the contraction principle?

(2) Let $g$ be a strictly monotone, continuously differentiable function. Show the change of variable formula

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(y) dy.$$ 

**Solution:** For every partition $\pi_0 = \{a = t_0, \ldots, t_n = b\}$, we may define $\tilde{\pi} = \{g(a), g(t_1), \ldots, g(t_n) = b\}$. Then we observe that

$$R_g(f \circ g, \pi, \xi) = \sum_k f g(\xi_k)[g(t_{k+1}) - g(t_k)] = R(f, \tilde{\pi}, g(\xi)).$$ 

Since $g'$ is bounded, $\text{mesh}(\pi) < \delta$ implies that $\text{mesh}(\tilde{\gamma}) < \|g'\|_{\infty} \delta$ (why). Thus by the definition of $I_g$ and the definition of the integral, we deduce that for every $\varepsilon > 0$ we can find $\delta > 0$ such that $\text{mesh}(\pi) < \frac{\delta}{1 + \|g'\|_{\infty}}$ we find

$$|I_g - I| \leq |I_g - R_g| + |R_g - R| + |R - I| \leq \varepsilon + \varepsilon \leq 2\varepsilon.$$ 

(3) Show that the space of $g$-integrable functions is linear.

**Solution:** As in the usual case we see that for positive $t$

$$U_g(f_1 + tf_2, \pi) \leq U_g(f_1, \pi) + t U_g(f_2, \pi)$$

and

$$L_g(f_1 + tf_2, \pi) \leq L_g(f_1, \pi) + t L_g(f_2, \pi).$$

As mentioned in class for $g$ integrable functions the limits $\inf_{\pi} U_g(f, \pi) = \sup_{\pi} L_g(f, \pi)$, and hence we have equality.

(4) Show that every monotone function is $g$ integrable. **Solution:** I recommend that we ignore this problem right now.
(5) Let \( d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases} \) be define on \( \mathbb{R} \). Show that the function \( f(x) = 1_{\mathbb{Q}}(x) \), the indicator function on the rational numbers now is continuous. Show that for every bounded sequence of function \( (f_n) \) on \( \mathbb{R} \) is equi-continuous with respect to this new distance function. What would Arzela-Ascoli mean in this modified scenario.

**Solution:** Let \( \varepsilon > 0 \) and define \( \delta = \frac{\varepsilon}{2} \). Then \( d(x, y) < \delta \) implies \( x = y \) and hence \( f(x) = f(y) \) for any function, in particular for \( f \) the indicator function this is true. The same argument applies for \( (f_n) \). Indeed, \( d(x, y) < \frac{1}{2} \) implies
\[
 f_n(x) = f_n(y)
\]
for all \( n \in \mathbb{N} \). Thus every bounded sequence is automatically equicontinuous. If \( \mathbb{R} \) with this funny distance were to be compact, we could conclude that
\[
 f(x) = \lim_{j} f_{n_j}(x)
\]
holds for a subsequence. That is certainly totally wrong. However, \( \mathbb{R} \) with this discrete metric is not compact.

(6) Can you find
(a) connected, non-compact sets?
(b) compact, non-connected sets?
(c) compact, not-path connected sets?

1) \( \mathbb{R} \) and 3) \( [0, 1] \cup [2, 4] \)

(7) Construct a sequence \( (f_k) \) of power series such that \( \lim_k f_k(x) = f(x) \) exists in \( \mathbb{R} \), but the derivative is not converging at some point.

**Solution:** Let \( f_k(x) = x^k e^{-(1+(x-1)^2^k)} \), great power series. \( \lim_k f_k(x) = 0 \) for all \( x \), and at \( x = 1 \), the sequence is constant. However,
\[
 f'_k(x) = k x^{k-1} e^{-(1+(x-1)^2^k)} + x^k e^{-(1+(x-1)^2^k)} (-2k)(x-1)^{2k-1} 
\]
satisfies \( f'_k(1) = k \) for \( k \geq 1 \).

(8) Let \( O \subset \mathbb{R} \) be an open set. A function \( u \) is called harmonic if
\[
 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]
Let \( D = \{(x, y) ||x|^2 + |y|^2 \leq 1 \} \). Assume that \( u : D \to \mathbb{R} \) is continuous and twice differentiable in \( \partial D = \{(x, y) ||x|^2 + |y|^2 < 1 \} \). Show that \( u \) attains its
minimum and maximum on the boundary $S^1 = \{(x, y) | x^2 + y^2 = 1\}$. **Hint:** Let $\varepsilon > 0$ and consider $v_\varepsilon(x, y) = u(x, y) + \varepsilon x^2 + \varepsilon y^2$.

**Solution:** Obviously $v_\varepsilon$ is continuous and attains its maximum $(x_0, y_0)$. Assume that happens in the interior. By considering $g(t) = v_\varepsilon(x_0 + t, y_0)$ we deduce $g''(t) \leq 0$ for a maximum. Thus $\frac{d^2 v_\varepsilon}{dx^2}(x_0, y_0) \leq 0$ and the same holds for the second $y$ partial. This implies

$$0 \geq \frac{d^2 v_\varepsilon}{dx^2} + \frac{d^2 v_\varepsilon}{dy^2} = \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + 4\varepsilon > 0$$

by assumption. Thus $(x_0, y_0)$ is on the boundary. Since $u$ is continuous we get

$$\sup_{(x,y)\in D} u(x, y) \leq \sup_{(x,y)\in D} v_\varepsilon(x, y) = \sup_{(x,y)\in \partial D} v_\varepsilon(x, y) \leq \sup_{(x,y)\in \partial D} u(x, y) + \sup_{(x,y)\in \partial D} \varepsilon x^2 + \varepsilon y^2 \leq \sup_{(x,y)\in \partial D} u(x, y) + \varepsilon .$$

Sending $\varepsilon \to 0$ yields the assertion.