1. practice problem for take home

(1) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be Lipschitz, i.e.

$$|F(x, y) - F(x', y')| \leq L(|x - x'|^2 + |y - y'|^2)^{1/2}.$$ 

For a fixed $g$ Consider the map $\Phi : C([0, \infty)) \to C([0, \infty))$

$$\Phi(f)(x) = g(x) + \int_0^x F(t, f(t))dt.$$ 

(a) Show that

$$|\Phi(f_1)(x) - \Phi(f_2)(x)| \leq \frac{L}{a} (e^{ax} - 1) \sup_t |f_1(t) - f_2(t)| e^{-at}.$$ 

(b) On $C([0, \infty))$ we use the distance

$$d_a(f_1, f_2) = \sup_x |f_1(x) - f_2(x)| e^{-ax}.$$ 

Show that for $a \geq 2L$

$$d_a(\Phi(f_1), \Phi(f_2)) \leq \frac{1}{2} d_a(f_1, f_2).$$ 

(c) Assume you could apply the contraction principle (or Banach Fixpoint Theorem) and obtain a fixpoint $\Phi(f) = f$. Find the differential equation satisfied by $f$.

(d) What is missing before you can apply the contraction principle?

(2) Let $g$ be a strictly monotone, continuously differentiable function. Show the change of variable formula

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(y)dy.$$ 

(3) Show that the space of $g$-integrable functions is linear.

(4) Show that every monotone function is $g$ integrable.

(5) Let $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ be define on $\mathbb{R}$. Show that the function $f(x) = 1_{\mathbb{Q}}(x)$, the indicator function on the rational numbers now is continuous.

Show that for every bounded sequence of function $(f_n)$ on $\mathbb{R}$ is uniformly continuous with respect to this new distance function. What would Arzela-Ascoli mean in this modified scenario.

(6) Can you find

(a) connected, non-compact sets?

(b) compact, non-connected sets?
(c) compact, not-path connected sets?

(7) Construct a sequence $(f_k)$ of power

(8) Let $O \subset \mathbb{R}$ be an open set. A function $u$ is called harmonic if

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

Let $D = \{(x, y) ||x|^2 + |y|^2 \leq 1\}$. Assume that $u : D \rightarrow \mathbb{R}$ is continuous and twice differentiable in $\partial D = \{(x, y) ||x|^2 + |y|^2 < 1\}$. Show that $u$ attains its minimum and maximum on the boundary $S^1 = \{(x, y) ||x|^2 + |y|^2 = 1\}$. **Hint:** Let $\varepsilon > 0$ and consider $v_\varepsilon(x, y) = u(x, y) + \varepsilon x^2 + \varepsilon y^2$. 