Practice Problems

(1) Let \((x_n)\) be sequence in a metric space and \(x\) be a point. Assume that every subsequence of \((x_n)\) admits another subsequence that converges to \(x\). Show that \(\lim_{n} x_n = x\).

**Solution:** Assume that \(x\) is not the limit. Then there exists an \(\varepsilon > 0\) and for every \(n_0\) there exists a \(n_1 > n_0\) such that
\[
d(x_n, x) \geq \varepsilon.
\]
Let \(n_0 = 1\) and \(k_1 = n_1\). Then we use \(n_0 = k_1\) and find \(k_2 > k_1\), and effectively, we find increasing subsequence \((k_j)\) such that \(d(x_{k_j}, x) \geq \varepsilon\) for all \(j\). By assumption we have a further subsequence such that \(\lim x_{k_{jm}} = x\) and hence eventually \(d(x, x_{k_{jm}}) < \varepsilon\). Contradiction.

(2) Let \((x_n) \subset K\), \(K\) compact \(x \in K\) be sequence and a point such that every subsequence admits another subsequence, and if that subsequence happens to converge, then the limit is \(x\). Show that \(\lim_{n} x_n = x\).

**Solution:** This follows from the previous problem because in a compact space every sequence has a convergent subsequence.

(3) Let \(K\) compact set and \(p \notin K\) be a point. Show that there exists a \(\delta > 0\) such that
\[
K_\delta \cap B(x, \delta) = \emptyset.
\]
Here \(K_\delta = \{y \in Y : \exists x \in K : d(x, y) < \delta\}\). This set is actually open, why?

**Solution:** \(K_\delta\) is open: Let \(d(x, y) < \delta\) for some \(x \in K\). Let \(\varepsilon \delta - d(x, y)\).

Let \(d(z, y) < \varepsilon\). Then \(d(z, x) < \delta\) by triangle inequality and hence
\[
B(y, \varepsilon) \subset K_\delta.
\]

Now for the main problem. For every \(y \in K\) we define \(\delta_y = d(x, y)\). Then
\[
K \subset \bigcup_y B(y, \delta_y/3).
\]
by compactness \(K\) is contained in \(N\) of such balls. Let \(3\delta = \min\{\delta_y|1 \leq j \leq N\}\). Let \(z \in K\). Find \(y_j\) such that \(d(y_j, z) < \delta_y/3\). Then
\[
\delta_y = d(x, y_j) \leq d(y_j, z) + d(z, x) \leq \frac{\delta_y}{3} + d(z, x)
\]
implies
\[
d(z, x) > \frac{2}{3}\delta_y.
\]
Similarly, \( d(w, z) < \delta \) implies
\[
\frac{2}{3} \delta y_j \leq d(z, x) \leq d(w, z) + d(w, x) \leq \frac{1}{3} \delta y_j
\]
and hence \( w \in K_\delta \) implies
\[
\delta \leq \frac{\delta y_j}{3} < d(w, x).
\]
This means \( B(x, \delta) \cap K_\delta = \emptyset \).

(4) Find an example of a sequence of continuous functions \((f_n)\), \( f \) such that \( \lim_n f_n(x) = f(x) \), and the \((f_n)\) do not converge uniformly to \( f \).

**Solution:** Let
\[
f_n(x) = \begin{cases} 
2nx & 0 \leq x \leq \frac{1}{2n} \\
1 - 2n(x - \frac{1}{2n}) & \frac{1}{2n} \leq x \leq \frac{1}{n} \\
0 & x \geq \frac{1}{n}
\end{cases}
\]
Clearly, \( \lim_n f_n(x) = 0 \) for all \( x \). However,
\[
\|f_n - 0\|_\infty = 1
\]
for all \( n \). (Take \( x = \frac{1}{2n} \)). Thus no uniform convergence.

(5) Let \( F : (X, d) \to (X, d) \) be such that
\[
d(F(x), F(y)) \leq \lambda d(x, y)
\]
for some \( 0 < \lambda < 1 \). Show that \( x_n = F^{(n)}(x_0) \) (n-fold iteration) is convergent.

**Solution:** We prove by induction that
\[
d(x_{n+1}, x_n) = d(F(x_n), F(x_{n-1})) \leq \lambda d(x_{n-1}, x_{n-2}) \leq \lambda^2 d(x_{n-2}, x_{n-3}) \cdots \lambda^{n-1} d(x_1, x_0).
\]
Then, the geometric series implies
\[
d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{\infty} \lambda^{k-1} = \lambda^{n-1} \frac{1}{1-\lambda}.
\]
Thus \((x_n)\) is Cauchy.

(6) Let \( f(x) = e^{-x^2/2} \). Define \( f_k(x) = f\left(\frac{x}{k}\right) \) for \( k \geq 1 \) as a function on \([0, 1]\). Show that the family \( \{f_k | k \geq 1\} \) is totally bounded. **Solution:** Certainly \( 0 \leq f_k(x) \leq 1 \) and hence \((f_k)\) is bounded. On the other hand
\[
|(f^k)'(x)| \leq \frac{1}{k} |f'(x)| \leq \frac{1}{k}.
\]
The fundamental theorem, which we still have to prove shows that $|x - y| < \delta$ implies $|f^k(x) - f^k(y)| < \delta$. Thus our family is (uniformly) equicontinuous and Arzela-Ascoli applies.

(7) Let $A$ be a non-empty set in a metric space. Define

$$d_A(x) = \inf_{y \in A} d(x, y).$$

(a) Show that $d_A$ is equicontinuous.

(b) Let $A_1 \subset A_2 \subset$ and $f_j(x) = d_{A_j}$. Show that $f_j$ is pointwise convergent.

(c) Let $A_1 \supset A_2 \supset$ closed sets in $[0,1]$ and $f_j(x) = d_{A_j}(x)$. Show that $f_j$ are uniformly convergent.

(d) What can you say about the limit?

**Solution:**

a) We first claim that

$$|d_A(x) - d_A(x')| \leq d(x, x').$$

Indeed, let $x, x' \in X$ and $\varepsilon > 0$. Then there exists a $y \in A$ such that

$$d(x, y) - \varepsilon \leq d_A(x) \leq d(x, y).$$

This implies

$$d_A(x') \leq d(x', y) \leq d(x, x') + d(x, y) \leq d(x, x') + d_A(x) + \varepsilon.$$

Sending $\varepsilon$ to 0 shows that

$$d_A(x') \leq d(x, x') + d_A(x).$$

Similarly

$$d_A(x) \leq d(x, x') + d_A(x').$$

Now, immediately implies equicontinuity.

b) The sequence $f_j(x)$ is decreasing and hence convergent. c) the sequence $f_j(x)$ is now increasing. Let $A = \cap_j A_j$. Then

$$d_{A_j}(x) \leq d_A(x)$$

for all $x$. We claim $\lim_j d_{A_j}(x) = d_A(x)$. Indeed, let $x_j \in A_j$ such that

$$d(x, x_j) - 1/j \leq d_{A_j}(x) \leq d(x, x_j).$$

By passing to a subsequence we may assume that $\lim_j x_j = x_\infty$ exists. Then

$$d(x, x_\infty) = \lim_j d_{A_j}(x).$$
In particular, $d_{A_j}(x)$ is bounded from above and hence $f(x) = \sup_j f_j(x)$ is a well-defined function. We have

$$|f(x) - f(y)| \leq \lim_j |f_j(x) - f_j(y)| \leq |x - y|$$

by i). Thus $f$ is continuous. Now we show that $f_j$ converges to $f$ uniformly. Let $\varepsilon > 0$ and $x \in [0,1]$ we can find $j(\varepsilon, x)$ such that

$$f_j(x) \leq f(x) \leq f_j(x) + \frac{\varepsilon}{3}$$

holds for $j \geq j(\varepsilon, x)$. By compactness

$$[0,1] \subset B_{\varepsilon/3}(x_1) \cup \cdots \cup B_{\varepsilon/3}(x_m).$$

Then we can define $J = \max\{j(\varepsilon, x_1), \ldots, j(\varepsilon, x_m)\}$. Then for ever $x'$ we can find $x_k$ with $|x - x_k| < \varepsilon/3$ and we get

$$f(x) \leq f_j(x) + \frac{\varepsilon}{3} \leq f_j(x_k) + \frac{\varepsilon}{3} < f_j(x) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$  

Thus shows that $d_{\infty}(f, f_j) < \varepsilon$ and we are done.

(d) It turns out that $A = \bigcup_j A_j$ is not empty. Indeed, for every $x$ and $x_j$ from above, we see that $x_\infty = \lim_k x_{j_k}$ is in $A$, and hence by monotonicity

$$d_A(x) = \sup_j d_{A_j}(x) = d(x, x_\infty) \geq d_A(x).$$

Thus $d_A$ is the (uniform) limit.

(8) Let $C \subset C([0,1])$ (Here $C([0,1])$ is a metric space with respect to the $d_{\infty}$ metric). Which of the following statements is true

(a) $C$ is compact iff $C$ is closed and bounded.

(b) $C$ is compact iff $C$ is closed and totally bounded.

(c) $C$ is compact iff $C$ is closed and there exists a sequence of finite sets $F_n \subset C$ such that

$$\lim \sup_n d_{F_n}(f) = 0.$$  

**Solution:** a) is wrong (HW), b) is correct because $C([0,1])$ is complete and closed subsets of complete metric space are complete. c) is correct. One make sure $F_n$ are subset of $C$, by working with $\varepsilon/2$.

(9) Let $f : [0,1] \to [0,1]$ be continuous. Find

$$\inf_{0 \leq s \leq 1} |f(s) - s|.$$  

Solution: The answer is 0. If \( f(s) \leq s \) for all \( s \). Then \( f(0) = 0 \). If \( f(s) \leq s \) for all \( s \). Then \( f(1) = 1 \). If there exists \( s_0 < s_1 \) such that
\[
f(s_0) < s_0 < s_1 < f(s_1)
\]
we may define
\[
A = \{ s : \forall s_0 < t < s f(t) < t \}
\]
Let \( s = \sup A \). By continuity \( f(s) \leq s \). For every \( \epsilon > 0 \) there exists \( s < t_{\epsilon} < s + \epsilon \) such that
\[
f(t_{\epsilon}) \geq f(t_{\epsilon}).
\]
By continuity
\[
f(s) = \lim_{\epsilon \to 0} f(t_{\epsilon}) \geq \lim_{\epsilon} t_{\epsilon} = s.
\]
This implies \( f(s) = s \).