Practice Problems

(1) Let (x_n) be sequence in a metric space and x be a point. Assume that every subsequence of (x_n) admits another subsequence that converges to x. Show that $\lim_n x_n = x$.

Solution: Assume that x is not the limit. Then there exists an $\varepsilon > 0$ and for every n_0 there exists a $n_1 > n_0$ such that

$$d(x_n, x) \;\geq\; \varepsilon \,.$$

Let $n_0 = 1$ and $k_1 = n_1$. Then we use $n_0 = k_1$ and find $k_2 > k_1$, and effectively, we find increasing subsequence (k_j) such that $d(x_{k_j}, x) \;\geq\; \varepsilon$ for all j. By assumption we have a further subsequence such that $\lim x_{k_{jm}} = x$ and hence eventually $d(x, x_{k_{jm}}) < \varepsilon$. Contradiction.

(2) Let $(x_n) \subset K$, K compact $x \in K$ be sequence and a point such that every subsequence admits another subsequence, and if that subsequence happens to converge, then the limit is x. Show that $\lim_n x_n = x$.

Solution: This follows from the previous problem because in a compact space every sequence has a convergent subsequence.

(3) Let K compact set and $p \notin K$ be a point. Show that there exists a $\delta > 0$ such that

$$K_\delta \cap B(x, \delta) \;=\; \emptyset \,.$$

Here $K_\delta = \{y \in Y : \exists x \in K : d(x, y) < \delta\}$. This set is actually open, why?

Solution: K_δ is open: Let $d(x, y) < \delta$ for some $x \in K$. Let $\varepsilon\delta - d(x, y)$. Let $d(z, y) < \varepsilon$. Then $d(z, x) < \delta$ by triangle inequality and hence

$$B(y, \varepsilon) \subset K_\delta$$

Now for the main problem. For every $y \in K$ we define $\delta_y = d(x, y)$. Then

$$K \subset \bigcup_y B(y, \delta_y/3) \,.$$

by compactness K is contained in N of such balls. Let $3\delta = \min\{\delta_{y_j} | 1 \leq j \leq N\}$. Let $z \in K$. Find y_j such that $d(y_j, z) < \delta_{y_j} 3$. Then

$$\delta_{y_j} \;=\; d(x, y_j) \;\leq\; d(y_j, z) + d(z, x) \;\leq\; \frac{\delta_{y_j}}{3} + d(z, x)$$

implies

$$d(z, x) > \frac{2}{3}\delta_{y_j} \,.$$

Similarly, $d(w,z) < \delta$ implies

$$\frac{2}{3}\delta y_j \;\le\; d(z,x) \;\le\; d(w,z) + d(w,x) \;\le\; \frac{1}{3}\delta_{y_j}$$

and hence $w \in K_\delta$ implies

$$\delta \;\le\; \frac{\delta_{y_j}}{3} < d(w,x)\;.$$

This means $B(x,\delta) \cap K_\delta = \emptyset$.

(4) Find an example of a sequence of continuous functions (f_n), f such that $\lim_n f_n(x) = f(x)$, and the (f_n) do not converge uniformly to f.

Solution: Let

$$f_n(x) \;=\; \begin{cases} 2nx & 0 \le x \le \frac{1}{2n} \\ 1 - 2n(x - \frac{1}{2n}) & \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & x \;\ge\; \frac{1}{n} \end{cases} .$$

Clearly, $\lim_n f_n(x) = 0$ for all x. However,

$$\|f_n - 0\|_\infty \;=\; 1$$

for all n. (Take $x = \frac{1}{2n})$). Thus no uniform convergence.

(5) Let $F : (X,d) \to (X,d)$ be such that

$$d(F(x),F(y)) \le \lambda d(x,y)$$

for some $0 < \lambda < 1$. Show that $x_n = F^{(n)}(x_0)$ (n-fold iteration) is convergent.

Solution: We prove by induction that

$$d(x_{n+1},x_n) \;=\; d(F(x_n),F(x_{n-1})) \;\le\; \lambda d(x_{n-1},x_{n-2}) \;\le\; \lambda^2 d(x_{n-2},x_{n-3})\cdots\lambda^{n-1}d(x_1,x_0)\;.$$

Then, the geometric series implies

$$d(x_m,x_n) \;\le\; \sum_{k=n}^{m-1} d(x_{k+1},x_k) \;\le\; \sum_{k=n}^{\infty} \lambda^{k-1} \;=\; \lambda^{n-1}\frac{1}{1-\lambda}\;.$$

Thus (x_n) is Cauchy.

(6) Let $f(x) = e^{-x^2/2}$. Define $f_k(x) = f(\frac{x}{k})$ for $k \;\ge\; 1$ as a function on $[0,1]$. Show that the family $\{f_k | k \;\ge\; 1\}$ is totally bounded. **Solution:** Certainly $0 \le f_k(x) \le 1$ and hence (f_k) is bounded. On the other hand

$$|(f^k)'(x)| \;\le\; |\frac{1}{k}f'(x)| \;\le\; \frac{1}{k}\;.$$

The fundamental theorem, which we still have to prove shows that $|x-y| < \delta$ implies $|f^k(x)-f^k(y)| < \delta$. Thus our family is (uniformly) equicontinuous and Arzela-Ascoli applies.

(7) Let A be a non-empty set in a metric space. Define

$$d_A(x) \ = \ \inf_{y\in A} d(x,y)\ .$$

(a) Show that d_A is equicontinuous.
(b) Let $A_1 \subset A_2 \subset$ and $f_j(x) = d_{A_j}$. Show that f_j is pointwise convergent.
(c) Let $A_1 \supset A_2 \supset$ closed sets in $[0,1]$ and $f_j(x) = d_{A_j}(x)$. Show that f_j are uniformly convergent.
(d) What can you say about the limit?

Solution: a) We first claim that

$$|d_A(x) - d_A(x')| \ \leq \ d(x,x')\ . \tag{0.1}$$

Indeed, let $x, x' \in X$ and $\varepsilon > 0$. Then there exists a $y \in A$ such thar

$$d(x,y) - \varepsilon \ \leq \ d_A(x) \ \leq \ d(x,y)\ .$$

This implies

$$d_A(x') \ \leq \ d(x',y) \ \leq \ d(x,x') + d(x,y) \ \leq \ d(x,x') + d_A(x) + \varepsilon\ .$$

Sending ε to 0 shows that

$$d_A(x') \ \leq \ d(x,x') + d_A(x)$$

Similarly

$$d_A(x) \ \leq \ d(x,x') + d_A(x')\ .$$

Now, immediately implies equicontinuity.

b) The sequence $f_j(x)$ is decreasing and hence convergent. c) the sequence $f_j(x)$ is now increasing. Let $A \ = \ \cap_j A_j$. Then

$$d_{A_j}(x) \ \leq \ d_A(x)$$

for all x. We claim $\lim_j d_{A_j}(x) = d_A(x)$. Indeed, let $x_j \in A_j$ such that

$$d(x,x_j) - 1/j \ \leq \ d_{A_j}(x) \ \leq \ d(x,x_j)\ .$$

By passing to a subsequence we may assume that $\lim_j x_j = x_\infty$ exists. Then

$$d(x,x_\infty) \ = \ \lim_j d_{A_j}(x)\ .$$

In particular, $d_{A_j}(x)$ is bounded from above and hence $f(x) = \sup_j f_j(x)$ is a well-defined function. We have

$$|f(x) - f(y)| \;\leq\; \lim_j |f_j(x) - f_j(y)| \;\leq\; |x - y|$$

by i). Thus f is continuous. Now we show that f_j converges to f uniformly. Let $\varepsilon > 0$ and $x \in [0,1]$ we can find $j(\varepsilon, x)$ such that

$$f_j(x) \;\leq\; f(x) \;\leq\; f_j(x) + \frac{\varepsilon}{3}$$

holds for $j \;\geq\; j(\varepsilon, x)$. By compactness

$$[0,1] \subset B_{\varepsilon/3}(x_1) \cup \cdots \cup B_{\varepsilon/3}(x_m) \,.$$

Then we can define $J = \max\{j(\varepsilon, x_1), ..., j(\varepsilon, x_m)$. Then for ever x' we can find x_k with $|x - x_k| < \varepsilon/3$ and we get

$$f(x) \;\leq\; f(x_j) + \frac{\varepsilon}{3} \;\leq\; f_J(x_k) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < f_J(x) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \,.$$

Thus shows that $d_\infty(f, f_J) < \varepsilon$ and we are done.

d) It turns out that $A = \bigcup_j A_j$ is not empty. Indeed, for every x and x_j from above, we see that $x_\infty = \lim_k x_{j_k}$ is in A, and hence by monotonicity

$$d_A(x) \;=\; \sup_j d_{A_j}(x) \;=\; d(x, x_\infty) \;\geq\; d_A(x) \,.$$

Thus d_A is the (uniform) limit.

(8) Let $C \subset C([0,1])$ (Here $C([0,1])$ is a metric space with respect to the d_∞ metric). Which of the following statements is true
  (a) C is compact iff C is closed and bounded.
  (b) C is comapct iff C is closed and totally bounded.
  (c) C is compact iff C is closed and there exists a sequence of finite sets $F_n \subset C$ such that

$$\lim_n \sup_{F \in C} d_{F_n}(f) \;=\; 0 \,.$$

Solution: a) is wrong (HW), b) is correct because $C([0,1])$ is complete and closed subsets of complete metric space are complete. c) is correct. One make sure F_n are subset of C, by working with $\varepsilon/2$.

(9) Let $f : [0,1] \to [0,1]$ be continuous. Find

$$\inf_{0 \leq s \leq 1} |f(s) - s| \,.$$

Solution: The answer is 0. If $f(s) \leq s$ for all s. Then $f(0) = 0$. If $f(s) \leq s$ for all s. Then $f(1) = 1$. If there exists $s_0 < s_1$ such that

$$f(s_0) < s_0 < s_1 < f(s_1)$$

we may define

$$A \;=\; \{s : \forall_{s_0 < t < s} f(t) < t\;.\}$$

Let $s = \sup A$. By continuity $f(s) \leq s$. For every $\varepsilon > 0$ there exists $s < t_\varepsilon < s + \varepsilon$ such that

$$f(t_\varepsilon) \;\geq\; f(t_\varepsilon)\;.$$

By continuity

$$f(s) \;=\; \lim_{\varepsilon \to 0} f(t_\varepsilon) \;\geq\; \lim_{\varepsilon} t_\varepsilon \;=\; s\;.$$

This implies $f(s) = s$. ■