Metric spaces

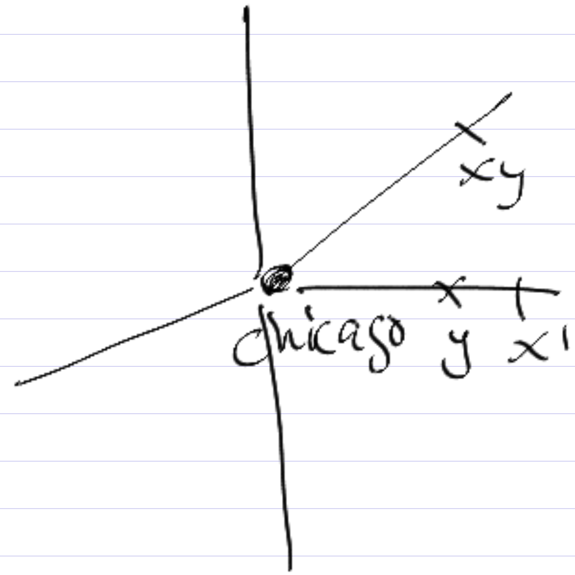
Def (X, d) is a metric space if

$d: X \times X \to [0, \infty)$ satisfies

i) $d(x,y) \geq 0$

ii) $d(x,y) = 0 \iff x = y$

iii) $d(x,y) \leq d(x,z) + d(z,y)$



Ex $d(x,y) = |x-y|$

$$d(x,y) = \left(\sum_{1}^{n} |x_j - y_j|^q\right)^{1/q} \quad 1 \leq q \leq \infty$$

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

$$d(x,y) = \begin{cases} |x-x'| & \text{on same ray} \\ d(x,p) + d(p,y) & \text{on different rays} \end{cases}$$

travelling salesmen.

Df $(x_n) \subseteq X$ is called Cauchy if

$$\forall \varepsilon > 0 \ \exists n_0 \ \forall n, m > n_0 \quad d(x_n, x_m) < \varepsilon$$

Df (x_n) is called convergence if $\exists x \in X$

$$\lim_{n \to \infty} d(x_n, x) = 0$$

Df (X, d) is called complete if every Cauchy sequences is convergent.

Ex $X = \mathbb{R}$

$X = (\mathbb{R}^n \ d_p)$

$X = (\mathbb{R}^2, d_{\text{chicago}})$ (HW)

$(X, \text{discrete})$ is always complete.

Def $O \subset X$ open $\forall x \in O \ \exists \varepsilon > 0 \ \mathrm{Ball}(x,\varepsilon) \subset O$

$\mathrm{Ball}(x,\varepsilon) = \{ y \mid d(x,y) < \varepsilon \}$.

Def $f: X \to Y$ ct at x_0 if

$\forall \varepsilon > 0 \ \exists \delta > 0 \quad f(\mathrm{Ball}(x_0,\delta)) \subseteq \mathrm{Ball}(f(x_0),\varepsilon)$

Lemma $f\ X \to Y$ ct at all $x_0 \iff f^{-1}(O)$ open $\forall O$ open

Proof $\Rightarrow$ $O \subset Y$ open $f(x_0) = y \in O$ $B(y,\varepsilon) \subseteq O$

$\exists \delta \quad B(x_0,\delta) \subseteq f^{-1}(O) \Rightarrow f^{-1}(O)$ open.

$\Leftarrow$ $B(y,\varepsilon)$ open $f(x_0) = y$ $\exists \delta > 0 \ B(x_0,\delta) \subset f^{-1}(B(y,\varepsilon))$ $\blacksquare$

<u>Remarks</u> $C \subseteq (X,d)$ closed $\underset{X \text{ complete}}{\Rightarrow}$ C complete.

<u>Remark</u> $C \subseteq (X,d)$ closed iff $\lim_n x_n = x \in X$ ex $(x_n) \subseteq C$ $\Rightarrow x \in C$.

(sequentially closed)

(HW)

Ex $(x_n) \subseteq X$

$$C = \{ x \in X \mid \exists \lim x_{n_k} = x \}$$

$$= \{ x \in X \mid \forall m \ \{ n \in \mathbb{N} \mid dist(x_n, x) < 1/m \} \text{ is infinite} \}$$

$$C^c = \{ x \in X \mid \exists m \ \{ n \in \mathbb{N} \mid dist(x_n, x) < 1/m \} \text{ finite} \}$$

Then $\quad \operatorname{dist}(x', x) < 1/2m$

$$\{ n \in \mathbb{N} \mid \operatorname{dist}(x', x_n) < 1/2m \}$$

$$\subseteq \{ n \in \mathbb{N} \mid \operatorname{dist}(x_n, x) < 1/m \} \quad \text{finite}$$

$\Rightarrow \quad x' \in \mathcal{O}.$ $\blacksquare$