Final Exam Practice Problems

(1) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable function such that $|F'| \le L$ on $[a,b] \times [c,d]$. Show that F is Lipschitz, i.e.

$$|F(x,y) - F(x',y')| \le L(|x-x'| + |y-y'|) .$$

Solution: We did this in class. Let

$$z(t) = (x + t(x'-x), y + t(y'-y))$$

and

$$\phi(t) = F(z(t))$$

Then by FT

$$\begin{aligned} |\phi(1) - \phi(0)| &= |\int F'(z(t))z'(t)dt| \\ &= |\int_0^1 \frac{dF}{dx}(z(t))(x'-x) + \frac{dF}{dy}(z(t))(y'-y)dt| \\ &\le \sup_z \max\{|\frac{dF}{dx}(z(t)|, |\frac{dF}{dy}(z(t)|\}(|x-x'| + |y'-y|) \end{aligned}$$

Thanks to the upper Darboux estimate.

(2) Let $f, g : [0,1] \to [1,\infty]$ be integrable. Show that $h(x) = f(x)^{g(x)}$ is integrable.

Solution: We may assume that f, g are bounded, i.e. $f(x) \in [1,d]$, $g(x) \in [1,d]$. The function

$$F(x,y) = x^y = e^{y \ln x}$$

is differentiable on $[1,d]^2$ and hence Lipschitz. This there exits a constant such that

$$|F(a,b) - F(a',b')| \le L(|a-a'| + |b-b'|) .$$

For a partition we define

$$\Delta_{I_j}(h) = \sup_{x,y \in I_j} h(x) - h(y) .$$

Then we easily prove that

$$\Delta_{I_j}(F(f,g)) \le L[\Delta_{I_j}(f) + \Delta_{I_j}(g)] .$$

For $\varepsilon > 0$ we can find a partitions π_f, π_g such that

$$\Delta_{\pi_f}(f) = \sum^{\pi}(f) - \sum_{\pi}(f) < \frac{\varepsilon}{2L}$$

and

$$\Delta_\pi(g) \;=\; \sum^{\pi}(g) - \sum_{\pi}(g) < \frac{\varepsilon}{2L} \,.$$

Take the partition $\pi = \pi^f \cap \pi^g$. Then we deduce for $h = F(f,g)$ that

$$\begin{aligned}\Delta_\pi(h) &= \sum_j |I_j|\Delta_{I_j}(h) \\ &\le L(\sum_j |I_j|\Delta_{I_j}(f) + \sum_j |I_j|\Delta_{I_j}(g)) \\ &\le\; L\Delta_{\pi_f}(f) + L\Delta_{\pi_g}(g) < \varepsilon \,.\end{aligned}$$

Here we use the monotonicity property of the Darboux integral difference expression. ■

(3) Let us recall that the remainder term in the Taylor formula is given by

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \varepsilon_2(f)(x)$$

where

$$\varepsilon_2(f)(x) \;=\; \int_{x_0}^{x} f''(s)(x-s) \,.$$

Show that if f_n and f_n' converge uniformly to f, f' on $[x_0 - 1, x_0 + 1]$, respectively, and f_n'', f'' are continuous, then $\varepsilon_2(f_n)$ converges uniformly to $\varepsilon_2(f)$.

Solution: We introduce

$$p_1^{f,x_0}(x) \;=\; f(x_0) + (x - x_0)f'(x_0) \,.$$

Note that pointwise convergence of f_n and pointwise convergence of f_n' at x_0 implies uniform convergence of

$$g_n(x) \;=\; p_1^{f_n,x_0}(x) \,.$$

However, thanks to the assumption

$$\varepsilon_2(f_n) \;=\; f_n - p_1^{f_n,x_0}$$

now converges uniformly on every compact interval. ■

(4) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be twice differentiable, and g be continuously differentiable. Find the derivative of

$$h(x) \;=\; \int_0^x F(x,y)g(y)dy \,.$$

Solution: Introduce

$$G(u,v) \;=\; \int_0^u F(v,y)dy\ .$$

Since F is assume to be continuous, we find

$$\frac{dG}{du} \;=\; F(v,u)$$

thanks to (second) FT. Now, we have to differentiate in the v direction and consider, for fixed u

$$h_n(y) \;=\; n(F(v+\frac{1}{n},y)-F(v,y)) - \frac{dF}{dv}(v,y) \;=\; \frac{dF}{dv}(\xi_n,y) - \frac{dF}{dv}(v,y)\ .$$

Here we use MVT and ote that $v \le \xi_n \le v+1/n$. By uniform continuity we deduce that

$$|h_n(y)| \;\le\; \varepsilon$$

for $n \;\ge\; n$. This h_n converges to 0 uniformly and we deduce

$$\lim_n n[G(v+\frac{1}{n},u)-G(v,u)] \;=\; \int_0^u \frac{dF}{dx}(v,y)dy\ .$$

Since we have continuity of these derivatives in v and u, we find

$$h'(x) \;=\; F(x,x) + \int_0^x \frac{dF}{dx}(x,y)dy\ .$$

■

(5) Show that the set

$$\{(x,z)| -a \le x \le a, \exists y : \cos^2(x+y) = z\}$$

is compact.

Solution: Since we (finally) restricted x to be in $[-a,a]$ we only need to consider $y \in [-a-2\pi, a+2\pi] = [-b,b]$. Then we define

$$\Phi(x,y) \;=\; (x, \cos^2(x+y))\ .$$

This is map for $[-a,a]\times[-b,b] \;\to\; \mathbb{R}^2$ and obviously continuous. Since $[-a,a]\times[-b,b]$ is compact the assertion follows by a well-known compactness preserving theorem from class. ■

(6) Let

$$B \;=\; \{f \in C[0,1]\ :\ \|f\| \le 1, f(0) = 1/2\}$$

Show that for every t

$$\{\int_0^1 (f(x)-t)^2 dx | f \in B\}$$

is a closed interval.

Solution: It is easy to see that the set

$$S \;=\; \{f \;:\; \|f\|_\infty \le 1, f(0) = 1/2\}$$

is convex. For fixed t we define

$$\Phi(f) \;=\; \int_0^1 (f(x) - t)^2 dx \;.$$

It suffices to show that Φ is continuous.

$$\begin{aligned}
|\Phi(f) - \Phi(g)| \;&\le\; \int |(f(x) - t)^2 - (g(x) - t)^2| dx \\
&\le\; \int |f(x)^2 - g(x)^2| dx + 2t \int |f(x) - g(x)| dx \\
&=\; \int |f(x) - g(x)||f(x)| + |g(x)| dx + 2t \int |f(x) - g(x)| dx \\
&\le \|f - g\|_\infty(\|f\|_\infty + \|g\|_\infty) + 2t\|f - g\|_\infty \\
&\le \|f - g\|_\infty(\|f - g\|_\infty + 2\|g\|)_\infty + 2t\|f - g\|_\infty \;.
\end{aligned}$$

Thus for fixed g, $\|f - g\|$ small implies that $|\Phi(f) - \Phi(g)|$ small. Since connectedness is preserved under continuous maps, we deduce the assertion. By the way calculating these sets may be a little tricky, and is definitely not part of the problem. Moreover, since we know that the range is an interval we only have to find min and max, a typical problem in control theory or PDE or physics.