(1) Let $X$ be a finite set and $d$ be a metric. Show that $(X, d)$ is complete.
(2) Let \((x_n)\) be a sequence in a metric space such that

\[ d(x_n, x_{n+1}) \leq 16 \times 3^{-n} \]

Show that \((x_n)\) is Cauchy. (Give details, it is not enough to refer to lecture).
(3) Let \((X,d)\) be a metric space and \(f : X \to \mathbb{R}\) be a function. Show that the following conditions are equivalent

i) For every \(r \in \mathbb{R}\) \(f^{-1}(r, \infty)\) is open;

ii) For every convergent sequence \(\lim x_n = x\)

\[ f(x) \leq \liminf x_n. \]
(1) Let $X$ be a finite set and $d$ be a metric. Show that $(X, d)$ is complete.

**Solution:** Let $(x_n)$ be a Cauchy sequence. Define $d = \min\{d(x, y) | x \neq y\}$ which is finite because $X$ is finite. Let $\varepsilon = \frac{d}{2}$ and $n_0$ such that $N > n_0$ implies $d(x_n, x_m) < \varepsilon$. Then $x_n = x_m$ by construction and $\lim_n x_n = x_{n_0+1}$.

(2) Let $(x_n)$ be a sequence in a metric space such that $d(x_n, x_{n+1}) \leq 16 \times 3^{-n}$

Show that $(x_n)$ is Cauchy. (Give details, it is not enough to refer to lecture).

**Solution:** We observe that for $n < m$

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} 163^{-k} \leq 163^{-n} \frac{1}{1 - 1/3} = 24 \times 3^{-n}.$$  

Thus for $\varepsilon > 0$ we choose $n_0$ such that $3^n > \frac{24}{\varepsilon}$ and find $d(x_n, x_m) < \varepsilon$ for all $n_0 < n < m$.

(3) Let $(X, d)$ be a metric space and $f : X \to \mathbb{R}$ be a function. Show that the following conditions are equivalent

i) For every $r \in \mathbb{R}$ $f^{-1}(r, \infty)$ is open;

ii) For every convergent sequence $\lim_n x_n = x$

$$f(x) \leq \liminf_n f(x_n).$$

i) $\Rightarrow$ ii). Let $\lim_n x_n = x$ and $r = f(x) - \varepsilon$. Then $O = f^{-1}(r, \infty)$ is open and hence for $n \geq n_0$ we have $x_n \in O$. This means

$$\inf_{n \geq n_0} \geq r - \varepsilon.$$  

Therefore

$$\liminf_n f(x_n) = \sup_{n_0} \inf_n f(x_n) \geq r - \varepsilon$$

for all $\varepsilon$. Sending $\varepsilon \to 0$ yields the assertion.
$ii \Rightarrow i)$. Let $r \in \mathbb{R}$. Assume that $O = f^{-1}(r, \infty)$ is not open. Then there exists an $x \in O$, that means $f(x) > r$ and a sequence $x_n \notin O$ such that

$$\lim_{n} x_n = x$$

By assumption

$$f(x) \leq \liminf_{n} f(x_n).$$

This means for some subsequence $\lim_{j} f(x_{n_j})$ we

$$f(x) \leq \lim_{j} f(x_{n_j}) \leq r$$

because $f(x_n) \leq r$ for all $n$. This means

$$f(x) \leq r < f(x).$$

This contradiction completes the proof.