1. Completion

FACT 1.1. 1) Let \((X,d)\) be a complete metric space. Then every closed subset is complete.

2) Let \(A \subset (X,d)\) be a set. The closure
\[ \tilde{A} = \bigcap_{C \subset \mathbb{C}} C \]
is a closed set and \(A\) is dense in \(\tilde{A}\). i.e for every \(y \in \tilde{A}\), \(\varepsilon > 0\) there exists \(x \in A\) such that
\[ d(x, y) < \varepsilon. \]

LEMMA 1.2. Let \(X\) be a non-empty metric space. For \(x \in X\), we define the function \(d_x : X \to \mathbb{R}\) by
\[ d_x(y) = d(y, x). \]

Then
i) \(d_x\) is continuous.

ii) \(d_\infty(d_x, d_z) = d(x, z)\) holds for
\[ d_\infty(f, g) = \sup_{y \in X} |f(y) - g(y)|. \]

PROOF. Part i) is a homework. For part ii) we note that
\[ |d_x(y) - d_z(y)| = |d(x, y) - d(z, y)| \leq d(x, z) \]
Indeed, we have
\[ d(x, y) \leq d(x, z) + d(z, y) \]
and
\[ d(z, y) \leq d(z, y) + d(y, x) \]
This implies
\[ d_\infty(d_x, d_z) \leq d(x, z) \]
Moreover, taking \(y = z\)
\[ d_\infty(d_x, d_z) \geq |d_x(z) - d_z(z)| = d(x, z). \]

We have shown ii).

THEOREM 1.3. The uniform limit of continuous functions is continuous.

PROOF. See notes or text book.
In the next corollary we may think of \( h \) to be a profile.

**Corollary 1.4.** Let \( h \in C(X, Y) \) a fixed function and 
\[
Z_h = \{ f : X \to Y | f \text{ continuous }, d_\infty(f, h) < \infty \}.
\]
Assume that \( Y \) is complete. Then \( Z_h \) is complete.

**Proof.** Let \( f_n \) by Cauchy in \( Z_h \). For every \( \varepsilon > 0 \) there exists a \( n_0 \) such that
\[
d(f_n(x), f_m(x)) < \varepsilon/2
\]
holds for all \( x \) and \( n, m \geq 0 \). In particular \( (f_n(x)) \) is Cauchy in \( Y \), and hence
\[
f(x) = \lim_n f_n(x)
\]
makes sense. By continuity, we have
\[
d(f_n(x), f(x)) < \varepsilon/2 < \varepsilon
\]
for all \( n \geq n_0 \) and \( x \). This means
\[
d_\infty(f_n, f) < \varepsilon
\]
for all \( n > n_0(\varepsilon) \). Therefore \( f_n \) is a uniform limit of continuous functions and hence continuous. Moreover, (1.2) shows that
\[
\lim_{n \to \infty} d_\infty(f_n, f) = 0.
\]
Therefore \( f \) is indeed, the correct limit. Finally, by choosing \( \varepsilon = 1 \) we find
\[
d_\infty(f, h) \leq d_\infty(f, f_{n_0(1)+1}) + d_\infty(f_{n_0(1)+1}, h) < \infty.
\]
This means our limit is indeed in \( Z_h \).

**Theorem 1.5.** Let \( \emptyset \neq X \) be a metric space. Then \( X \) gas a completion, i.e. there exists a metric space \( Y \) and a map \( \phi : X \to Y \) such that
\begin{enumerate}
  
  i) \( d(\phi(x), \phi(y)) = d(x, y) \);
  
  ii) \( Y \) is complete;
  
  iii) \( \phi(X) \) is dense in \( Y \).
\end{enumerate}

**Proof.** Let \( x_0 \in X \) and \( h = d_{x_0} \) our profile in \( C(X, \mathbb{R}) \). We define
\[
\phi(x) = d_x
\]
assigning every point its distance function. Then \( Z_h \) is complete. Moreover,
\[
d_\infty(d_x, d_{x_0}) \leq d(x, x_0)
\]
1. COMPLETION

shows that $\phi(X) \subset Z_h$. Then the closure $Y = \overline{\phi(X)}$ is complete and $\phi(x) = d_x$ has all the desired properties.