1. Completion

Fact 1.1. *1) Let (X, d) be a complete metric space. Then very closed subset is complete.*
2) Let $A \subset (X, d)$ be a set. The closure

$$\bar{A} = \bigcap_{A \subset C} C$$

is a closed set and A is dense in $\bar{A}$. i.e f or every $y \in \bar{A}$, $\varepsilon > 0$ there exists $x \in A$ such that

$$d(x, y) < \varepsilon .$$

Lemma 1.2. *Let X be a non-empty metric space. For $x \in X$, we define the function $d_x : X \to \mathbb{R}$ by*

$$d_x(y) = d(y, x) .$$

Then

i) *d_x is continuous.*
ii) *$d_\infty(d_x, d_z) = d(x, z)$ holds for*

$$d_\infty(f, g) = \sup_{y \in X} |f(y) - g(y)| .$$

Proof. Part i) is a homework. For part ii) we note that

$$|d_x(y) - d_z(y)| = |d(x, y) - d(z, y)| \le d(x, z)$$

Indeed, we have

$$d(x, y) \le d(x, z) + d(z, y)$$

and

$$d(z, y) \le d(z, y) + d(y, x) .$$

This implies

$$d_\infty(d_x, d_z) \le d(x, z) .$$

Moreover, taking $y = z$

$$d_\infty(d_x, d_z) \ge |d_x(z) - d_z(z)| = d(x, z) .$$

We have shown *ii*). ■

Theorem 1.3. *The uniform limit of continuous functions is continuous.*

Proof. See notes or text book. ■

In the next corollary we may think of h to be a profile.

COROLLARY 1.4. *Let $h \in C(X,Y)$ a fixed function and*

$$Z_h = \{f : X \to Y | f \, continuous \, , \, d_\infty(f,h) < \infty\} \, .$$

Assume that Y is complete. Then Z_h is complete.

PROOF. Let f_n by Cauchy in Z_h. For every $\varepsilon > 0$ there exists a n_0 such that

$$d(f_n(x), f_m(x)) < \varepsilon/2 \tag{1.1}$$

holds for all x and $n, m \geq 0$. In particular $(f_n(x))$ is Cauchy in Y, and hence

$$f(x) = \lim_n f_n(x)$$

makes sense. By continuity, we have

$$d(f_n(x), f(x)) \leq \varepsilon/2 < \varepsilon$$

for all $n \geq n_0$ and x. This means

$$d_\infty(f_n, f) < \varepsilon \tag{1.2}$$

for all $n > n_0(\varepsilon)$. Therefore f_n is a uniform limit of continuous functions and hence continuous. Moreover, (1.2) shows that

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0 \, .$$

Therefore f is indeed, the correct limit. Finally, by choosing $\varepsilon = 1$ we find

$$d_\infty(f,h) \leq d_\infty(f, f_{n_0(1)+1}) + d_\infty(f_{n_0(1)+1}, h) < \infty \, .$$

This means our limit is indeed in Z_h. ∎

THEOREM 1.5. *Let $\emptyset \neq X$ be a metric space. Then X gas a completion, i.e. there exists a metric space Y and a map $\phi : X \to Y$ such that*

i) *$d(\phi(x), \phi(y)) = d(x,y)$;*
ii) *Y is complete;*
iii) *$\phi(X)$ is dense in Y.*

PROOF. Let $x_0 \in X$ and $h = d_{x_0}$ our profile in $C(X, \mathbb{R})$. We define

$$\phi(x) = d_x$$

assigning every point its distance function. Then Z_h is complete. Moreover,

$$d_\infty(d_x, d_{x_0}) \leq d(x, x_0)$$

shows that $\phi(X) \subset Z_h$. Then the closure $Y = \overline{\phi(X)}$ is complete and $\phi(x) = d_x$ has all the desired properties. ■