Descartes circle method

Idea: The tangent line can be found by looking at double roots of a circle intersecting the graph.

**USUALLY: 2 SOLUTIONS**
In order to obtain the numerical value, Descartes has to find the equations which describe the circle. The midpoint is given by \(a+v\) on the x-axis. Since \(f(a)\) is on the circle we have

\[y^2 + (x-(a+v))^2 = f(a)^2 + v^2\]

Solutions are therefore given by

\[f(x)^2 + (x-a)^2 - 2(x-a)v + v^2 = f(a)^2 + v^2\]

or

\[f(x)^2 - f(a)^2 + (x-a)^2 - 2(x-a)v = 0\]

We have to choose \(v\) such that there is a double solution for \(x=a\)!

Example: \(f(x)=x^2\)

\[x^4 - a^4 + (x-a)^2 - 2(x-a)v = 0\]

and \(a\) is a double root. Note that

\[x^4 - a^4 = (x-a) (x^3 + x^2a + xa^2 + a^3)\]

Therefore

\[(x^3 + x^2a + xa^2 + a^3) + (x-a) - 2v = 0\]

implies \(4a^3 - 2v = 0\),

\[v = -2a^3\]
\( T = \nabla f \)
\[ y^2 + w^2 = f(a)^2 + (a - (\lambda + w)) \]
\[ = f(x)^2 \]
\[ p(x) - p(a)^2 = \frac{2}{x-a} \cdot (x-a) \]

\[ = 2w(x-a) - (x-a)^2 \]

\[ \frac{p(x) - p(a)^2}{x-a} = 2w - (x-a) \]

\[ \text{Hence} \quad 2w = g(a, a) \]

\[ \text{Note} \quad g(a, a) = 2p'(x)f(x) \]

\[ \therefore w = \frac{p'(x)f(x)}{2} \]

poly

\[ \text{Using modern techniques} \]

\[ x = a \]
Final step:

\[ y = f(x) \]

\[ \text{slope} \parallel \frac{-f'(a)}{VW} \]

\[ \left(-\frac{VW}{f'(a)}\right) = \left(\frac{f'(a)}{W}\right) \]

\[ \text{slope} \quad \frac{VW}{f'(a)} = \frac{2a}{a^2} = 2a \]

\[ \Rightarrow f'(a) \]
ANOTHER EXAMPLE:

\[ f(x) = \sqrt{x} \]

\[ x - a + (x-a)^2 + 2v(x-a) = 0 \]

Double 0:

\[ 1 + (x-a) - 2v = 0 \]

\[ v = \frac{1}{2} \]

\[ f'(a) = \frac{1}{2\sqrt{a}} \]

METHOD:

1. PRODUCE EQUATION
2. DIVIDE BY \((x-a)\)
3. SET \(x = a\)
4. SOLVE FOR \(v\) \[ \frac{w}{f'(x)} \]