Leibniz (1646-1716)

Background:
- entered university at age 15,
- bachelor 17
- starts math at age 26
- diplomatic mission in Paris
- prime time for math
- influence of Huygens, Pascal
- proud self-educator
- learned geometry later
- wrote his own memoirs
- describes his own discovery
- day time job
  - administrator for
    - Hannover sovereign
- achievements in logic, philos, physics

Goal/Dream: The correct setup will make calculations reliable, and feasible for all people not just the experts.

Keeps his naive perspective.
Starting small:

Discovered an inverse relation between sums and differences.

Remark: \[ d_1 + \ldots + d_n = a_1 \quad \text{or} \quad a_{n+1} \]

Example: \((i+1)^2-i^2=2i+1\). Hence \(n^2-0^2=(2n-1)+(2n-3)+\ldots+1\) the sum of odd numbers.

Note: \(2n+(2n-2)+\ldots+2+0=(2n-1)+(2n-3)+\ldots+1+n\)

Hence \(2n+\ldots+1=2n^2+n=n(2n+1)\)

Leibniz did similar tricks with \(n^3!\)

Leibniz also applied this to infinite series (assuming that \(a_n\) converges to 0) and his harmonic triangle (instead of pascal triangle suggested by Huygens).

General \(\sum \Delta a_n = \frac{1}{2} \Delta a_{n+1} = \frac{1}{2} a_{n+1} - a_n\)
Characteristic triangle

Following a script of Pascal (Dattonville) Leibniz discovered and generalized characteristic triangles to a change of variable formula.

Starting point

\[ \triangle \theta \]

\[ \triangle \theta \]

\[ A \quad I \]

\[ \triangle \theta \]

\[ AID \cong \triangle KE_2 \]

are similar
General curve

Application: Body of revolution

Note: \( n^2 = (y^2 + nu^2) \) and \( nu/y = dy/dx \)

\[
\frac{ds}{dx} = \frac{n}{y} \quad \text{(suital)}
\]

\[
yds = ndx
\]

\[
n = y \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}
\]
Example:
\[ y = f(x) \]
then
\[ n = f(x)(1 + f'(x)^2)^{1/2} \]

For example \( f(x) = x^{1/2} \) gives
\[ n = x^{1/2}(1 + 1/4x)^{1/2} \]
\[ = \frac{1}{2} (4x + 1)^{1/2} \]
Therefore the area obtained from revolving the parabola around the x-axis is given by

\[
\text{Surf.-Area} = \int_0^1 \sqrt{1 + y'^2} \, ds
\]
\[ = 2\pi \int_0^1 n \, dx \]
\[ = \pi \left[ \frac{2}{3} \left(4x + 1\right)^{3/2} \right]_0^1 \]
\[ = \frac{\pi}{3} \left( \frac{2^{3/2}}{3} - 1 \right) \]
Application 2: Rectification

\[ \triangle \text{similar to } ABC \]

Hence \( \frac{ds}{dy} = \frac{t}{a} \)

\( \alpha \) \( \alpha ds = t dy \)
This means the arclength calculation boils down to an area calculation, provided we can get a good handle on \( t = t(y) \)

Example: \( y = x^{2/3} \), \( dy/dx = 2/3x^{-1/3} \) we get (\( a = 1 \))

\[
t = ds/dy = (dx^2 + dy^2)^{1/2}/dy
= ((dx/dy)^2+1)^{1/2}
= (9/4x^{2/3}+1)^{1/2}
= (9/4y+1)^{1/2}
\]

Therefore for \( x \) between 0 and 8 we find \( y \) between 0 and 4 and hence

\[
L = \int_0^4 \left(1 + \frac{2}{3}y\right)^{1/2} dy
= \frac{4}{3} \left(1 + \frac{2}{3}y\right)^{3/2} \bigg|_0^4
= \frac{3}{2^7} \left(10^{3/2} - 1\right)
\]
Application 3:

\[ \frac{dy}{dx} = \frac{b}{y} \]

Hence
\[ \int_{0}^{5} \int_{0}^{y} dy \, dx = \frac{b^2}{2} \]

if starting at origin

Hence we can integrate functions if they occur as nu of a function with \( f(0) = 0 \).
Illustration: Hudde and Sluse calculated
\[ nu = mx^{2m-1} \] for \( y = x^m \)

Therefore
\[
\int_{a}^{b} \frac{2m-1}{2m} x^{2m-1} \, dx = \frac{a^m}{2m} - \frac{b^m}{2m}
\]

Note \( m \) need not be an integer!

Page 243: Leibniz obtained not really new results, but tools towards his goal:Algorithmic calculus.