Eudoxus

Eudoxus of Chios (408-355BC?), a student of Plato's academy, became the greatest mathematician of the fourth century BC.

One of his major contributions was a key definition which made it possible to compare geometric magnitudes which itself were not rational. It is important to note that Eudoxus' definition allows us to show that certain relations are the same, without assuming that compared objects are the same: For example square of radii and areas of circles.

**Eudoxos definition:**
Let a, b be comparable magnitudes and c, d be comparable magnitudes. Then
\[(a:b) = (c:d)\]
if
\[na > mb \text{ and } nc > md\]
or
\[na = mb \text{ and } nc = md\]
or
\[na < mb \text{ and } nc < md\]
holds for all natural numbers n, m.

**My version of Eudoxos definition:**
\[(a:b) = (c:d) \text{ if and only if }\]
\[\begin{align*}
na &> mb \\
nc &> md
\end{align*}\]
In Eudoxos definition two objects $a,b$ are comparable if we
i) in principle can decide wether $na>b$
ii) There always exists an integer such that
$na>b$

The book calls ii) Eudoxus axiom. In Math 444 or 447 you will
call this Archimedean axiom.

**Lemma:** Let $a,b,c$ be comparable. Then

$$(a:c) = (b:c) \implies a=b$$

**Proof:**

1) **My def:** Assume $a>b$. Then

$$a>b \implies c>c$$

Similarly $a<b \implies c<c$.  

2) **Ed. Def:** Assume $a>b$. Then $(n=1=m)$

$$a>b \text{ and } c>c \text{ or } a<b \text{ and } c<c \text{ or } a=b \text{ and } c=c$$

3) **Proof (book):** Assume $a>b$. By i) there exists $n$ st

$$n(a-b)>c$$

Let $mc$ such that

$$mc > nb \geq (m-1)c$$

We get $na > nb + c > (m-1)c + c = mc$ and

$$nb < mc$$

This contradicts Eud. def. if $(a:c) = (b:c)$
Equipped with Eudoxus definition rigorous proofs of elementary facts on area are possible and are executed in Euclid's elements. In fact logical rigor and the art or arguing mathematically is developed on the way!

The book observes that with the help of Eudoxos definition one can introduce the Eudoxian semigroup of magnitudes (such as areas) which don't have to be numbers, but satisfy

- Associativity \( a+(b+c)=(a+b)+c \)
- Commutativity \( a+b=b+a \)
- Monotonicity \( a>b \) implies \( a+c>b+c \)
- EA there exists an integer such that \( na>b \).
The greeks considered different magnitudes which could be compared: areas, diameter, circumference, volumes of elementary objects formed, for example, by polygons. Although some of these magnitudes were discovered to be incomparable, magnitudes of the same type followed the following rules

1) (Montonicity) If S is contained in T, then a(S) is smaller or equal to a(T).
2) (Additivity) If R is the non-overlapping union of two objects S and T, then
   \[ a(R) = a(S) + a(T) \]
3) Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude grater than its half, and if this process is repated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let \( a(T) > a(S) \). Then

\[ \frac{a(T)}{2^n} \leq a(S) \]

for some \( n \) eventually (Eudoxusian axiom)
Remark:

With the help of this exhaustion axiom many rigorous results can be proved.

Theorem:

Let C be a circle. Let Pn be regular polygons inscribed in the circle, so that in every step the number of points are doubled. Then the remaining area can be made arbitrary small.

Lemma:

Let M_n be the difference area 
\[ a(C) - a(P_n) \]
Then 
\[ M_{n+1} \leq \frac{1}{2} M_n \]

Proof: We will show 
\[ M_n - M_{n+1} \geq \frac{1}{2} M_n \]
For this we fix a sector, and add a middle line.
Then
\[ M_n = n \cdot D_n \quad M_{n+1} = n \cdot D_{n+1} \quad \text{and} \]
\[ D_n - D_{n+1} = \frac{1}{2} a \left( \triangle \right) \]
\[ = \frac{1}{2} a \left( \square \right) \]
\[ > \frac{1}{2} a \left( \bigcirc \right) \]
\[ = \frac{1}{2} D_n \]
Thus \[ M_n - M_{n+1} > \frac{1}{2} D_n \]
Theorem: \((a(C_D): a(C_d)) = (D^2 : d^2)\)

Proof: Let \(n, m \in \mathbb{N}\) and

\[n \ a(C_D) > m \ a(C_d)\]

We can find an inscribed polygon such that

\[n \ a(C_D) > n \ a(P_D^k) > m \ a(C_d)\]

Thus

\[n \ a(P_D^k) > m \ a(P_d^k)\]

Since \((a(P_D^k) : a(P_d^k)) = (D^2 : d^2)\), we get

\[n D^2 > m d^2\]
Let us now assume
\[ \text{na}(C_D) < \text{ma}(C_D) \]

Then we find regular polygons with
\[ \text{na}(C_D) < \text{ma}(P^k_d) \]
Thus \[ \text{na}(P^k_d) < \text{ma}(P^k_d) \]
hence \[ nD^2 < mD^2 \]

**Remark:** The book explains a double reductio ad absurdum argument, which handles Eudoxus' case
\[ \text{na}(C_D) = \text{ma}(C_D) \]
**THEOREM (EUDOXUS)**

The volume of a pyramid is one third of the product of base area and height.

**LEMMA:**

A pyramid can be partitioned into two pyramids \( P_1 \) and \( P_2 \) with half of the height and a quarter of the base area and two prism \( Q_1 \) and \( Q_2 \) with the same property such that

\[
\text{vol}(Q_1) + \text{vol}(Q_2) > \text{vol}(P_1) + \text{vol}(P_2)
\]

**Proof:** Every line is split in \( \frac{1}{2} \)

We can move \( P_1 \) in \( Q_2 \) and \( P_2 \) in \( Q_1 \). This gives volume as intended.
Lemma:

Let A be the area of the base of the original pyramid and H be its height. The volumes satisfy
\[
\text{vol}(Q_1) = \text{vol}(Q_2) = \frac{A}{4} \cdot \frac{H}{2} = \frac{AH}{8}
\]

Hence
\[
\text{vol}(Q_1 + Q_2) = \frac{AH}{8}
\]

Theorem (Eudoxus): Let P and P' be pyramids of the same height and base areas A, A'. Then
\[
\left( \frac{\text{vol}(P)}{\text{vol}(P')} \right) = \left( \frac{A}{A'} \right)
\]

Proof:

Let n, m be integers such that
\[
nA < mA'
\]
Assume that
\[
n \text{vol}(A) > m \text{vol}(A')
\]

Then we perform the splitting Lemma iteratively and focus on the union of prisms we obtain at each step. In each step the volume of the prism capture more than half of the original volume and hence the difference volume between the pyramid and the union of the prism can be made arbitrarily small. Note that we get 2 prims in the first step, then splitting the two remaining pyramids we get another 4, and so on. Thus we can find a union of prisms U_k and U_k' such that
\[
n \text{vol}(A) > n \text{vol}(U_k) > m \text{vol}(A')
\]
We get
\[ n \text{ vol}(U_k) > m \text{ vol}(U_k') \quad (\&) \]
However the heights of these prisms for P and P' is the same and the base area proportional to the base areas each step of the way. Therefore
\[ (A:A') = (\text{vol}(U_k):\text{vol}(U_k')) \]
and \( nA < mA' \) implies
\[ n \text{ vol}(U_k) < m \text{ vol}(U_k') \]
This contradicts (\&) and hence we have
\[ n \text{ vol}(P) \leq m \text{ vol}(P') \]
We have proved
\[ nA < mA' \text{ implies } n \text{ vol}(P) \leq m \text{ vol}(P') \]
With a similar argument we can prove
\[ nA > mA' \text{ implies } n \text{ vol}(P) \geq m \text{ vol}(P') \]
and this concludes the proof.