In general it is not easy to calculate the length of a curve!
Latin name: Rectification
In the late 1650s infinitesimal techniques were applied:

\[ \Delta S = \sqrt{\Delta x^2 + \Delta y^2} \]

Applying this formula for an arbitrary curve we obtain:

\[ S = \sum (\Delta x_i^2 + \Delta y_i^2)^{\frac{1}{2}} = \sum (1 + (\frac{dy}{dx})^2) \Delta x_i \]
For example we may consider the Archimedean spiral:
Here we have

\[ b = \frac{2\pi a}{h} \]

\[ \text{Single} \quad (n+1) b \]

\[ D = \sqrt{\left(\frac{a+h}{2}\right)^2 + H^2} \]
\[ H = \left(1 - \cos \frac{2\pi}{2n}\right) b i \]
\[ \alpha \approx b i \quad \beta \approx b(i+1) \]

\[ \Delta S \equiv \sqrt{(b_1)^2 + (b_1)(1 - \cos \frac{2\pi}{2n})^2} \]

\[ \equiv b c \sqrt{1 + (1 - \cos \frac{2\pi}{2n})^2} \]

**Therefore**

**Arc Length**

**Given**

\[ \sum \Delta S_i \]
Better example by Williams (≈ 20)
in 1650: \( y^3 = x^2 \)

\[ \Delta s_1 = \sqrt{\Delta y_1^2 + \Delta x_1^2} \]

\[ \Delta y_1 = x^{3/2}_{t+1} - x^{3/2}_1 \]

\[ = \frac{3}{2} \int_0^{x_{t+1}} t^{1/2} \, dt - \frac{3}{2} \int_0^{x_1} t^{1/2} \, dt \]

\[ = \frac{3}{2} A_{t+1} - A_t \]

Introduce \( z = \sqrt{x} \)

\[ \cong \frac{3}{2} z_1 \Delta x_1 \]

\[ \Rightarrow \Delta s_1 = \sqrt{1 + \frac{9}{4} z_1^2} \Delta x_1 \]
The total length is then given by

\[ s = \sum_{i} \sqrt{1 + \left( \frac{x_i}{\Delta x_i} \right)^2} \Delta x_i = \frac{3}{2} \sqrt{\frac{2}{3}} \left( \left( \frac{4}{5} + q \right)^{3/2} - \left( \frac{4}{5} \right)^{3/2} \right) = \left( \frac{4}{5} + q \right)^{3/2} - \left( \frac{4}{5} \right)^{3/2} \]
Remark: Thomas Hariot came up with an argument for the equiangular spiral before Neil!

This can be a project
Tangent constructions

Fermat used an argument which we would now call `variational principle' in order to determine the maximum of $x(b-x)$
This gives the maximal area of a square with fixed parameter.
Let us assume the function is maximal at \( x \). Then at \( x+e \) we have

\[
b(x+e) - (x+e)^2 = bx + be - x^2 - 2xe - e^2
\]

\[
= bx - x^2 + be - 2xe - e^2
\]

\[
\approx bx - x^2
\]

\[
\Rightarrow be - 2xe - e^2 = 0
\]

\[
\Rightarrow b = 2x \quad \text{(ignore)}
\]
Drawback: This argument could work if \( e \) were not 0 but infinitesimal small.

Fermat does not provide a more detailed explanation!

The book calls this pseudo calculus. At any rate. This method works in order to calculate tangent lines:

\[
\text{Similar } \Delta \Rightarrow \frac{s + e}{s} = \frac{k}{f(x)} \\
k \sim f(x + \epsilon) \text{ gives}
\]
\[ 1 + \frac{e}{5} = \frac{p(x + e)}{p(x)} \]

\[ \Rightarrow \quad \frac{e}{5} = \frac{p(x + e) - p(x)}{p(x)} \]

\[ S = \frac{e \cdot p(x)}{p(x + e) - p(x)} \]

In the limit, this gives

\[ S = \frac{f(x)}{f(x)} \]
In his notation

\[ f(x) = x^2 \]

\[ S = \frac{f(x)}{f(x+e) - f(x)} \]

\[ = \frac{x^2}{(x+e)^2 - x^2} \cdot e \]

\[ = \frac{x^2}{x^2 + 2ex + e^2 - x^2} \cdot e \]

\[ = \frac{x^2}{2ex + e} \approx \frac{x}{2} \]

\[ S = x^2 \Rightarrow f'(x) = \frac{p(x)}{S} = \frac{x^2}{x^2} = \frac{x}{x^2} = \frac{1}{x} \]

\[ S = 2x \]
Descartes circle method

Idea: The tangent line can be found by looking at double roots of a circle intersecting the graph.

\[ x^2 + 4 \]

**USUALLY: 2 SOLUTIONS**
In order to obtain the numerical value, Descartes has to find the equations which describe the circle. The midpoint is given by a+v on the x-axis. Since f(a) is on the circle we have

\[ y^2 + (x-(a+v))^2 = f(a)^2 + v^2 \]

Solutions are therefore given by

\[ f(x)^2 + (x-a)^2 - 2(x-a)v + v^2 = f(a)^2 + v^2 \]

or

\[ f(x)^2 - f(a)^2 + (x-a)^2 - 2(x-a)v = 0 \]

We have to choose \( v \) such that there is a double solution for \( x=a \)!

Example: \( f(x)=x^2 \)

\[ x^4 - a^4 + (x-a)^2 - 2(x-a)v = 0 \]

and \( a \) is a double root. Note that
\[ x^4 - a^4 = (x-a)(x^3 + x^2a + xa^2 + a^3) \]

Therefore
\[ (x^3 + x^2a + xa^2 + a^3) + (x-a)-2v = 0 \]

implies \( 4a^3 - 2v = 0 \),

\[ v = -2a^3 \]
Final step:

\[ y = f(x) \]

\[ \text{slope} \parallel -\frac{f'(a)}{\sqrt{V}} \]

\[ \left( \frac{V}{f'(a)} \right) \bigg| \left( \frac{f(a)}{V} \right) \]

\[ \text{slope} = \frac{\sqrt{V}}{f'(a)} = \frac{\sqrt{2a^3}}{a^2} = 2a \sqrt{7} \]
ANOTHER EXAMPLE:

\[ f(x) = \sqrt{x} \]

\[ x - a + (x-a)^2 + 2\sqrt{(x-a)} = 0 \]

Double \( O \):

\[ 1 + (x-a) - 2\sqrt{V} = 0 \]

\[ V = \frac{1}{4} \]

\[ f'(a) = \frac{1}{2\sqrt{a}} \]

METHOD:

1. Produce equation
2. Divide by \((x-a)\)
3. Set \(x = 0\)
4. Solve for \(V\)