Euclid’s Elements
Book X
Proposition 36

If two rational straight lines commensurable in square only are added together, then the whole is irrational; let it be called binomial.

Definition 1: Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure.

Definition 2: Straight lines are commensurable in square when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure.

Definition 3: Let then the assigned straight line be called rational, and those straight lines which are commensurable with it, whether in length and in square, or in square only, rational, but those that are incommensurable with it irrational.

For us, AB will be the assigned straight line that we call rational. So if a line is commensurable to AB it is rational and if a line is incommensurable to AB it is irrational.

AB and BC are commensurable in square only so: \( AB^2 = p \cdot BC^2 \) (but \( AB \neq p \cdot BC \)) where \( p \) is a rational number.

\[
\begin{align*}
AB &= AB \cdot BC \\
BC &= BC \cdot BC
\end{align*}
\]

since AB is incommensurable with BC

\( AB \cdot BC \) is incommensurable with \( BC^2 \) \hspace{1cm} (Book X Prop. 11)

\( 2(AB \cdot BC) \) is commensurable with \( (AB \cdot BC) \) \hspace{1cm} (Definition 1)

\( (AB^2 + BC^2) \) is commensurable with \( BC^2 \) \hspace{1cm} (Book X Prop. 15)

\[2 \, (AB \ast BC) \text{ is incommensurable with } AB^2 + BC^2 \quad \text{(Book X Prop. 13)}\]

\[AC^2 = 2(AB \ast BC) + (AB^2 + BC^2) \quad \text{(Book 2 Prop. 4)}\]

\[AC^2 \text{ is incommensurable with } (AB^2 + BC^2) \quad \text{(Book X Prop. 16)}\]

\[AB^2 + BC^2 \text{ is a rational number: By Proposition 15, } AB^2 + BC^2 \text{ is commensurable to } BC^2. \text{ By Definition 3, } BC^2 \text{ is commensurable with the assigned line } AB, \text{ since } AB \text{ and } BC \text{ are commensurable in square only. Therefore } AB^2 + BC^2 \text{ is commensurable to } AB \text{ so } AB^2 + BC^2 \text{ is rational.}\]

\[AC^2 \text{ is incommensurable with } (AB^2 + BC^2) \text{ which is commensurable with } AB. \text{ By Proposition 13 } AC^2 \text{ is incommensurable with } AB, \text{ therefore } AC^2 \text{ is irrational. Therefore we know } AC \text{ is irrational:}\]

Assume \(AC = pAB\), then \(AC^2 = p^2AB^2\) which is a contradiction since \(AC^2\) is incommensurable with \(AB^2\).

\(AC\) is irrational so Proposition 36 holds.
**Book X Proposition 11:** If four magnitudes are proportional, and the first is commensurable with the second, then the third is also commensurable with the fourth; but, if the first is incommensurable with the second, then the third also is incommensurable with the fourth.

\[ \frac{A}{B} = \frac{C}{D} \]

Let \( A \) be commensurable to \( C \) and \( B \) be incommensurable to \( D \). By assumption \( A = pC \) but \( B \neq pD \) which would imply \( A \neq B \) a contradiction.

**Book X Proposition 15:** If two commensurable magnitudes are added together, then the whole is also commensurable with each of them; and, if the whole is commensurable with one of them, then the original magnitudes are also commensurable.

Say two commensurable magnitudes \( A \) and \( B \) are added together. Since \( A = pB \), \( A + B = pB + B = (p+1)B \) which is commensurable with \( B \). Use the same argument to prove commensurability with \( A \).
**Book X Proposition 13:** If two magnitudes are commensurable, and one of them is incommensurable with any magnitude, then the remaining one is also incommensurable with the same.

If \( A = pB \) and \( B \neq pC \)
Assume \( A \) is commensurable with \( C \), \( A = qC \). But \( A = pB \); so \( pB = qC \) which would make \( B \) and \( C \) commensurable, which is a contradiction.

**Book 2 Proposition 4:** If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.

\[
AC^2 = (AB + BC)^2 = AB^2 + BC^2 + 2(AB \cdot BC)
\]

\[
AB^2 = AB \cdot BC
\]

**Book X Proposition 16:** If two incommensurable magnitudes are added together, the sum is also incommensurable with each of them; but, if the sum is incommensurable with one of them, then the original magnitudes are also incommensurable.

Say \( AC^2 \) is commensurable with \( AB^2 + BC^2 \). Then \( AC^2 = p(AB^2 + BC^2) \). So

\[
p(AB^2 + BC^2) = 2(AB \cdot BC) + (AB^2 + BC^2) \quad \text{so} \quad 2(AB \cdot BC) = (p-1)(AB^2 + BC^2) \quad \text{which contradicts that \( AB \cdot BC \) is incommensurable with \( AB^2 + BC^2 \).}