Connection between differential forms and flux

We have seen surface integrals in two two different forms

$$\int_S F \cdot n dS$$

and

$$\int_S \omega \ .$$

Here

$$\omega \ = \ P\, dx_2 \wedge dx_3 + Q\, dx_3 \wedge dx_1 + R\, dx_1 \wedge dx_2$$

is a differential form.

We have also seen that the divergence theorem

$$\int_M div(F) dV \ = \ \int_{\partial M} F \cdot n dS$$

is a special case of the generalized Stokes theorem

$$\int_M d\omega \ = \ \int_{\partial M} \omega \ .$$

This means there has to be a very close connection between the two type of surface integrals. This is indeed the case.

Lemma 0.1. *Let $g : D \to g(D)$ be a parametrization of a surface $S = g(D)$. Then the non-normalized normal vector satisfies*

$$N(s,t) \ = \ \frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t} \ = \ \begin{pmatrix} \det g'_{23} \\ \det g'_{31} \\ \det g'_{12} \end{pmatrix} \ .$$

Proof. Let $x_1 = g_1(s,t)$, $x_2 = g_2(s,t)$ and $x_3 = g_3(s,t)$. Then we have

$$\begin{aligned} \frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t} &= \begin{pmatrix} \frac{\partial g_1}{\partial s} \\ \frac{\partial g_2}{\partial s} \\ \frac{\partial g_3}{\partial s} \end{pmatrix} \times \begin{pmatrix} \frac{\partial g_1}{\partial t} \\ \frac{\partial g_2}{\partial t} \\ \frac{\partial g_3}{\partial t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial g_2}{\partial s}\frac{\partial g_3}{\partial t} - \frac{\partial g_3}{\partial s}\frac{\partial g_3}{\partial t} \\ \frac{\partial g_3}{\partial s}\frac{\partial g_1}{\partial t} - \frac{\partial g_1}{\partial s}\frac{\partial g_3}{\partial t} \\ \frac{\partial g_1}{\partial s}\frac{\partial g_2}{\partial t} - \frac{\partial g_2}{\partial s}\frac{\partial g_1}{\partial t} \end{pmatrix} \end{aligned}$$

In order to prove the let us consider for example

$$g_{12}(s,t) \;=\; \begin{pmatrix} g_1(s,t) \\ g_2(s,t) \end{pmatrix}$$

Then we have

$$g_{12}' \;=\; \begin{pmatrix} \frac{\partial g_1}{\partial s} & \frac{\partial g_1}{\partial t} \\ \frac{\partial g_2}{\partial s} & \frac{\partial g_2}{\partial t} \end{pmatrix} .$$

This gives

$$\det g_{12}' \;=\; \frac{\partial g_1}{\partial s}\frac{\partial g_2}{\partial t} - \frac{\partial g_2}{\partial s}\frac{\partial g_1}{\partial t} .$$

The other terms work exactly the same way. □

Corollary 0.2. *Let G be an oriented surface in $\mathbb{R}^3$. Then*

$$\int_G P\,dx_2 \wedge dx_3 + Q\,dx_3 \wedge dx_1 + R\,dx_1 \wedge dx_2 \;=\; \int_G \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot n dS .$$

Proof. Let $g : D \to G$ be a parametrization. Then we deduce from the definition

$$\begin{aligned}
&\int_G P\,dx_2 \wedge dx_3 + Q\,dx_3 \wedge dx_1 + R\,dx_1 \wedge dx_2 \\
&= \int_D [P(g(s,t)) \det g_{23}'(st) + Q(g(s,t)) \det g_{31}'(st) + R(g(s,t)) \det g_{12}'(st)] dA(s,t) \\
&= \int_D \begin{pmatrix} P(g(s,t)) \\ Q(g(s,t)) \\ R(g(s,t)) \end{pmatrix} \cdot N(s,t) dA(s,t) \\
&= \int_G \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot n\,dS .
\end{aligned}$$

□

A special case is given for surfaces given by graphs

$$g(s,t) \;=\; \begin{pmatrix} s \\ t \\ h(s,t) \end{pmatrix} .$$

Here h is a differentiable height function. Then

$$N(s,t) = \frac{\partial g}{\partial s} \times \frac{\partial g}{\partial t} \;=\; \begin{pmatrix} 1 \\ 0 \\ \frac{\partial h}{\partial s} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial h}{\partial t} \end{pmatrix} \;=\; \begin{pmatrix} -\frac{\partial h}{\partial s} \\ -\frac{\partial h}{\partial t} \\ 1 \end{pmatrix} .$$

Connection between line integral and differential forms

For line integral we can apply a similar procedure, but this is significantly easier. For differential form of degree 1

$$\omega \;=\; Pdx_1 + Qdx_2 + Rdx_3 \,,$$

and a parametrization $g(t) \;=\; \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$, we find

$$\begin{aligned} \int_{\gamma([a,b])} \omega &= \int_a^b P(\gamma(t))\frac{\partial x_1}{\partial t}dt + Q(\gamma(t))\frac{\partial x_2}{\partial t}dt + R(\gamma(t))\frac{\partial x_3}{\partial t}dt \\ &= \int_{\gamma([a,b])} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot Tds \;=\; \int_{\gamma([a,b])} \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot dr \end{aligned}$$

Here dr is the derivative of the position vector r.
Therefore the general Stokes theorem

$$\int_{\partial G} \omega \;=\; \int_G d\omega$$

should be read as

$$\int_{\partial G} \vec{V} \cdot \vec{dr} \;=\; \int_G \mathrm{curl}(V) \cdot dS \;=\; \int_G \mathrm{curl}(V) \cdot n dS \,.$$

The right hand side is the flux of the curl. The curl appears naturally when we calculate

$$
\begin{aligned}
d\omega &= d(Pdx_1 + Qdx_2 + Rdx_3) \\
&= \frac{\partial P}{\partial x_1} dx_1 \wedge dx_1 + \frac{\partial P}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial P}{\partial x_3} dx_3 \wedge dx_1 \\
&\quad + \frac{\partial Q}{\partial x_1} dx_1 \wedge dx_2 + \frac{\partial Q}{\partial x_2} dx_2 \wedge dx_2 + \frac{\partial Q}{\partial x_3} dx_3 \wedge dx_2 \\
&\quad + \frac{\partial R}{\partial x_1} dx_1 \wedge dx_3 + \frac{\partial R}{\partial x_2} dx_2 \wedge dx_3 + \frac{\partial R}{\partial x_3} dx_3 \wedge dx_3 \\
&= (\frac{\partial R}{\partial x_2} - \frac{\partial Q}{\partial x_3}) dx_2 \wedge dx_3 + (\frac{\partial P}{\partial x_3} - \frac{\partial R}{\partial x_1}) dx_3 \wedge dx_1 + (\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}) dx_1 \wedge dx_2 \; .
\end{aligned}
$$

Note that we just learned above that for

$$\phi \;=\; V_1 dx_2 \wedge dx_3 + V_2 dx_3 \wedge dx_1 + V_3 dx_1 \wedge dx_2$$

we have

$$\int_G \phi \;=\; \int_G V \cdot n dS \; .$$

However, in our particular situation we have

$$V \;=\; \mathrm{curl} F \quad \text{if} \quad F \;=\; \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$$

is given by the three components as above.