Welcome to

CALCULUS III

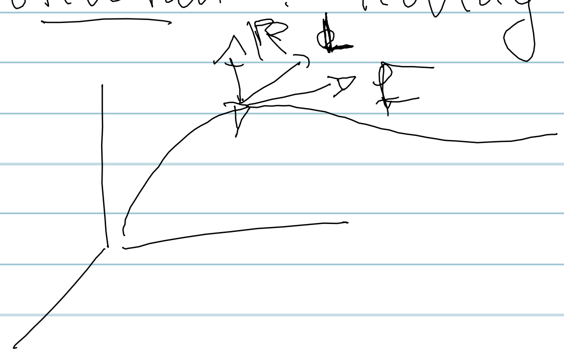
by Marius Junge
Altgeld 363

Previously Calculus was about

- Differentiation
- Integration
- Limits
- function in one variable

Now: Differentiation and Integration of functions of several variables

Motivation: Moving particle



Moving frame on roller coaster

F forward

L Left R right.

or

Plan

1) Vectors in 2D

2) Vectors in 3D

3) Differentiation

4) Integration

5) Combining the two
Fundamental Theorem
of calculus in disguise

Notation

$\mathbb{R}$ real line (e, $\sqrt{2}$, $\frac{p}{q}$, $k \in$)

$\mathbb{N}$ natural numbers ($0 \in \mathbb{N}$ or not ?)
1, 2, 3, 4 ....

$\mathbb{Z}$ integers $0, \pm k, k \in \mathbb{N}$

A set = collection of objects

A, B set

$$A \times B = \{ (a,b) : a \in A, b \in B \}$$

read as "the set of all pairs (a,b) such that a is in A, b is in B

$a \in A$ a is an element of A.

Example $A = \{ x \in \mathbb{R} : x \geq 0 \}$

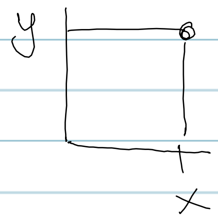
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x,y) : x \in \mathbb{R}, y \in \mathbb{R} \}$$

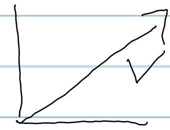
$$\mathbb{R}^d = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d \text{ times}} = \{ (x_1, \ldots, x_d) : x_1 \in \mathbb{R}, \ldots, x_d \in \mathbb{R} \}.$$

Vectors versus points

(moral distinction)

$\mathbb{R}^2 = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}\}$





$((0,0),(x,y))$

Physics: $p = (x,y)$ is a point or location.

v is the direction of going from origin to point.

vectors correspond to velocity of force

Remark one can add vectors

Definition: In a vector space (such as $\mathbb{R}^2$) one can vectors and stretch vectors

$$\lambda v + \mu w = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \mu \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $w = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$= \begin{pmatrix} \lambda x_1 + \mu y_1 \\ \lambda x_2 + \mu y_2 \end{pmatrix}$$

Note (usually $v = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ $w = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

$$\lambda \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \mu \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 + \mu x_2 \\ \lambda y_1 + \mu y_2 \end{pmatrix}$$)

Rules 1) $\lambda(v+w) = \lambda v + \lambda w$

2) $v + (-v) = 0$ where $-v$

$v = \begin{pmatrix} x \\ y \end{pmatrix}$ $-v = \begin{pmatrix} -x \\ -y \end{pmatrix}$

3) $v + (w+z)$

$= (v+w) + z$

4) $v + w = w + v$

Here $\lambda \in \mathbb{R}$ $v, w \in \mathbb{R}^2$ (actually $\mathbb{R}^d$)

HW Check that rules 1) to 4) hold

for function spaces

$V = \{ f: [0,1] \to \mathbb{R} \mid f \text{ continuous} \}$

$\lambda(f+g) = \lambda f + \lambda g$

where $(f+g)(t) = f(t) + g(t)$

and $(\lambda f)(t) = \lambda f(t)$

The spanning problem in 2D

Problem I Given two vectors v, w in $\mathbb{R}^2$.

When is it true that for all z in $\mathbb{R}^2$

we can find λ, μ such that

$$(*) \quad z = \lambda v + \mu w \quad ?$$

Examples 1) $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

2) $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3) $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $w = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

4) $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $w = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

Answer: If v, w do not point in same direction up to sign !

Def $\{v, w\}$ is called a basis for $\mathbb{R}^2$ if every point can be expressed by $(*)$

<u>Problem II</u> Let $\{v, w\}$ be a basis.

How can we find λ, μ?

$\# \quad z = \begin{pmatrix} x \\ y \end{pmatrix}$

Answer: Solve linear equation $v = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ $w = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$

$$x = \lambda a_{11} + \mu a_{21}$$

$$y = \lambda a_{21} + \mu a_{22}$$

<u>Examples</u>

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\lambda + \mu = 3$$
$$2\lambda + 3\mu = 7$$

$$\mu = 3 - \lambda$$

$$7 = 2\lambda + 3(3 - \lambda)$$
$$= 9 - \lambda$$

$$\lambda = 2 \quad \mu = 1.$$

General method

Definition A $n \times m$ matrix is a collection

1) of $n \cdot m$ real numbers

$$A = \begin{bmatrix} a_{ij} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & & & \\ & & & \\ a_{n1} & & & a_{nm} \end{bmatrix}$$

2) let $x = \begin{pmatrix} x_1 \\ \\ x_m \end{pmatrix}$ be a vector in m space.

Then $Ax = \begin{bmatrix} a_{11}x_1 + \cdots & + a_{1m}x_m \\ & \\ a_{n1}x_1 + \cdots & a_{nm}x_m \end{bmatrix}$

is a vector in n-space

3) One can multiply a $n \times m$ matrix with a $m \times k$ matrix and gets a $n \times k$ matrix

$$A \cdot B = \begin{bmatrix} a_{12} b_{11} + \cdots + a_{1m} b_{m1} & & \\ \vdots & \cdot & \cdot \\ a_{n1} b_{11} + \cdots a_{nm} b_{m1} & a_{n1} b_{m1} & a_{nm} b_{mk} \end{bmatrix}$$

How helpful?

Observation:

$$z = \lambda v + \mu w$$

means $z = a\begin{pmatrix}\lambda\\ \mu\end{pmatrix}$ $\quad a = (v\ w)$

If we can find c such that

$$ca = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ then:}$$

$$c(z) = ca\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\lambda\\ \mu\end{pmatrix} = \begin{pmatrix}\lambda\\ \mu\end{pmatrix}$$

Hence λ, μ are given by $c(z)$!

We traded ~~...~~ Problem II against

Problem III Find c with $ca = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$!

Note Pb II and Pb III are really equivalent (equally difficult) if we look for a general algorithm which works for all z (HW)

Key Example

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $0 \cdot f + ch = 1$ $\qquad h = \frac{1}{c}$

$\underbrace{0 \cdot e}_{=0} + cg = 0$ $\qquad g = 0$

$ae = 1$ $\qquad e = \frac{1}{a}$

$af + bh = 0$

$$f = -\frac{bh}{a} = -\frac{b}{ac}$$

Solution is

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{a^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}}$$

The inverse of a