Practice problems-Solutions

1) Let $a = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. In the following problems decide wether $\{v, w\}$ is a spanning system (linearly independent). If the answer is calculate the matrix with respect to the new system

i) $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;

ii) $v = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$, $w = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ for $0 \leq t \leq 2\pi$.

iii) $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

Solution: We calculate the determinant and find

$$\det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \, ,$$

$$\det \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t)1 & \cos(t) \end{pmatrix} = \cos^2(t) - \sin^2(t) \, ,$$

$$\det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1 \, ,$$

Thus in i) and ii) this systems are linearly independent. For ii) we find linear independence unless $\cos^2(t) = \sin^2(t)$ which happens for $t = \frac{\pi}{4} + \pi k$ with k a natural number. (Draw the functions for $|\sin(t)|$ and $|\cos(t)|$ which repeat each other period π).

2) Find the derivative

i) $f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2y \\ y^2x \end{pmatrix}$; $f' \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2xy \end{pmatrix}$

ii) $f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2y \\ y^2x \\ x^2y^2 \end{pmatrix}$; $f' \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2xy \\ 2xy^2 & 2x^2y \end{pmatrix}$;

iii) $f\begin{pmatrix} x \\ y \end{pmatrix} = xy^2 + yx^2$; $f'\begin{pmatrix} x \\ y \end{pmatrix} = (y^2 + 2xy, 2xy + x^2)$;

iv) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^2 + 2yx \\ x^2 + 2xy \\ x^2y^2 \end{pmatrix}$; $f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y & 2y + 2x \\ 2x + 2y & 2x \\ 2xy^2 & 2x^2y \end{pmatrix}$

v) Let $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix}$. Calculate g^{-1} and differentiate

$$F\begin{pmatrix} x \\ y \end{pmatrix} = g^{-1}fg$$

for f as in i). Here the solution is easy because

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

And hence the inverse function is given by multiplication with multiplication with the inverse $a^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$. Indeed,

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall that for any linear transformation $g_a(\vec{x}) = a\vec{x}$ we have $g_a' = a$. Thus we get $\bar{x} = x + y$, $\bar{y} = x - y$

$$\begin{aligned} F'(x, y) &= a^{-1}f'(g_a(x, y))a \\ &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 2\bar{x}\bar{y} & \bar{x}^2 \\ \bar{y}^2 & 2\bar{x}\bar{y} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 2\bar{x}\bar{y} + \bar{x}^2 & 2\bar{x}\bar{y} - \bar{x}^2 \\ \bar{y}^2 + 2\bar{x}\bar{y} & \bar{y}^2 - 2\bar{x}\bar{y} \end{pmatrix} \\ &= \begin{pmatrix} 2\bar{x}\bar{y} + \frac{\bar{x}^2+\bar{y}^2}{2} & \frac{\bar{y}^2-\bar{x}^2}{2} \\ \frac{\bar{x}^2-\bar{y}^2}{2} & -\frac{\bar{x}^2+\bar{y}^2}{2} + 2\bar{x}\bar{y} \end{pmatrix} = \end{aligned}$$

$$\begin{pmatrix} 2(x^2-y^2)+x^2+y^2 & -2xy \\ 2xy & -(x^2+y^2)+2(x^2-y^2) \end{pmatrix}$$

$$= \begin{pmatrix} 3x^2-y^2 & -2xy \\ 2xy & x^2-3y^2 \end{pmatrix}$$

I hope you got something similar!

vi) $f(x) = \cos(x)^{\sin(x^2)} = e^{\sin(x^2)\ln(\cos(x))}$. $f'(x) = \cos(x)^{\sin(x^2)}(2x\cos(x^2)\ln(\cos(x)) - \frac{\sin(x^2)\sin(x)}{\cos(x)}$. You can also do this with chain rule by considering

$$F(x,y) = x^y = e^{y\ln x} \quad F'(x,y) = (yx^{y-1}, x^y \ln x)$$

and $g(x) = \begin{pmatrix} \cos(x) \\ \sin(x^2) \end{pmatrix}$, $g'(x) = \begin{pmatrix} -\sin(x) \\ 2x\cos(x^2) \end{pmatrix}$ Then $f(x) = F(g(x))$ and

$$f'(x) = \sin(x^2)\cos(x)^{\sin(x^2)-1}(-\sin(x)) + \cos(x)^{\sin(x^2)}\ln\cos(x)(2x\cos(x^2)) .$$

The result is the same!

v) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2yx \end{pmatrix}$, $f'\begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} 2y & 2x \\ 0 & 2y \end{pmatrix} \begin{pmatrix} 2x & 0 \\ 2y & 2x \end{pmatrix} \right)$.

In a text book you will often find the row vector of the two Hessians (where f_1 is the function f from problem i))

$$f_1''(x,y) = \begin{pmatrix} \begin{pmatrix} 2y & 2x \\ 2x & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 2y \\ 2y & 2x \end{pmatrix} \end{pmatrix} .$$

3) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be functions such that

$$f(g\begin{pmatrix} r \\ \theta \end{pmatrix}) = \begin{pmatrix} r \\ \theta \end{pmatrix} .$$

Show that $f'(g(r,\theta)) = g'(r,\theta)^{-1}$. Apply this to $g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$.

Solution: Let f and g such that

$$f(g\begin{pmatrix} r \\ \theta \end{pmatrix}) = \begin{pmatrix} r \\ \theta \end{pmatrix} .$$

Then differentiation gives

$$f'(g(r,\theta))g'(r,\theta) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we calculate

$$g'(r,\theta) \;=\; \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}.$$

The inverse matrix is given by

$$g'(r,\theta)^{-1} \;=\; \frac{1}{r\cos^2(\theta)+r\sin^2(\theta)}\begin{pmatrix} r\cos(\theta) & r\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \;=\; \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{pmatrix}.$$

Thus we have (why)

$$f'(g(r,\theta)) \;=\; g'(r,\theta)^{-1}$$

and hence

$$f'(x,y) \;=\; \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}.$$

4) Let $r(t) = \begin{pmatrix} t \\ \frac{t^2}{2} \\ \frac{2\sqrt{2}}{3}t^{3/2} \end{pmatrix}$. Calculate the arclength, dr/ds, d^2r/ds^2 and the curvature.

Solution: We have $r'(t) = \begin{pmatrix} 1 \\ t \\ \sqrt{2t} \end{pmatrix}$, and hence

$$s(t) \;=\; \int_0^t \sqrt{1+u^2+2u}\,du \;=\; \int_0^t (1+u)du \;=\; t+\frac{t^2}{2}.$$

This gives $ds/dt = (1+t)$. Recall that

$$\frac{dr}{ds} \;=\; \frac{dr}{dt}\frac{dt}{ds} \;=\; \begin{pmatrix} \frac{1}{1+t} \\ \frac{t}{1+t} \\ \frac{\sqrt{2t}}{1+t} \end{pmatrix}.$$

An alternative way is to claculate t in terms of s. Indeed, we have

$$1+2s \;=\; t^2+2t+1 \;=\; (t+1)^2.$$

Therefore

$$t = \sqrt{1+2s} - 1 \,.$$

This gives

$$r(s) = \begin{pmatrix} \sqrt{1+2s} - 1 \\ 2 + 2s - 2\sqrt{1+2s} \\ \frac{2\sqrt{2}}{3}(\sqrt{1+2s} - 1)^{3/2} \end{pmatrix} .$$

And therefore

$$\frac{dr}{ds} = \begin{pmatrix} \frac{s}{\sqrt{1+2s}} \\ 2 - \frac{2s}{\sqrt{1+2s}} \\ \sqrt{2}(\sqrt{1+2s} - 1)^{1/2}\frac{s}{\sqrt{1+2s}} \end{pmatrix} .$$

Now calculating the second derivative is easy:

$$\frac{d^2r}{ds^2} = \begin{pmatrix} \frac{1}{\sqrt{1+2s}} - \frac{s^2}{(1+2s)^{3/2}} \\ \frac{2s^2}{(1+2s)^{3/2}} \\ \sqrt{2}(\frac{s^2}{1+2s} + (\sqrt{1+2s} - 1)^{1/2}(\frac{1}{\sqrt{1+2s}} - \frac{s^2}{(1+2s)^{3/2}}) \end{pmatrix} .$$

Finally the curvature κ is given by $|\frac{d^2r}{ds^2}|$, and that is just a matter of calculation.

5) In the following two problems you are given a vector function $g : [a, b] \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(g(t)) = c \,,$$

where c is a constant. This means g takes values in a level set of the surface $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \right\}$. Check whether<u>both</u> dg/dt and d^2g/d^2t lie in the tangent plane.

i) $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2$ and $c = 1$, $g(t) = \begin{pmatrix} r\cos(t) \\ r\sin(t) \\ \sqrt{1-r^2} \end{pmatrix}$ for some $0 \le r \le 1$.

ii) $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z - e^{-\sqrt{x^2+y^2}}$, $c = 0$ and $g(t) = \begin{pmatrix} r(t)\cos(t) \\ r(t)\sin(t) \\ e^{-r(t)} \end{pmatrix}$ such that $r = (r')^2$. (This means $r(t) = \frac{t^2}{4}$ will work with $r(0) = 0$, but there are more examples).

iii) Why is there no need to check that the dg/dt is tangent?

Solution: Let us start with iii): Assume that surface S is given by $f(x, y, z) = const$ and $g(t) \in S$. Then we get

$$f(g(t)) = const\,.$$

Differentiation gives

$$\nabla f(g(t)) \cdot g'(t) \;=\; f'(g(t))g'(t) \;=\; 0\,.$$

But the tangent plane is exactly defined by normal vector ∇f. Let us now discuss the examples. The first surface is a sphere and we have $z = \sqrt{1 - x^2 - y^2}$. The we decided to calculate the tangent plane as span of the vectors

$$v \;=\; \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix} \;=\; \begin{pmatrix} 1 \\ 0 \\ \frac{-x}{\sqrt{1-x^2-y^2}} \end{pmatrix}$$

and

$$w \;=\; \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 1 \\ \frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix}\,.$$

We observe that for our function g we have

$$g'(t) \;=\; \begin{pmatrix} -r\sin(t) \\ r\cos(t) \\ 0 \end{pmatrix}$$

and

$$g''(t) \;=\; \begin{pmatrix} -r\cos(t) \\ -r\sin(t) \\ 0 \end{pmatrix}\,.$$

Then

$$g''(t) \;=\; \lambda v + \mu w$$

implies $\lambda = -r\cos(t) = -x$, $\mu = -r\sin(t) = -y$. Thus $g''(t)$ lies in the tangent plane if

$$-x\frac{-x}{\sqrt{1-x^2-y^2}} - y\frac{-y}{\sqrt{1-x^2-y^2}} \;=\; 0\,.$$

Note, however, that the first term is

$$\frac{r^2}{\sqrt{1-r^2}}$$

Thus we need $r = 0$ and in that case $g(t)$ is constant. Now we go for ii) and we have $z = e^{-\sqrt{x^2+y^2}}$. Let us introduce $r = \sqrt{x^2+y^2}$. Then we have

$$v = \begin{pmatrix} 1 \\ 0 \\ -\frac{x}{r}e^{-r} \end{pmatrix}$$

and

$$w = \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{r}e^{-r} \end{pmatrix} .$$

Also we have

$$g'(t) = \begin{pmatrix} -r(t)\sin(t) + r'(t)\cos(t) \\ r(t)\cos(t) + r'(t)\sin(t) \\ -r'(t)e^{-r(t)} \end{pmatrix}$$

and

$$g''(t) = \begin{pmatrix} -r(t)\cos(t) - 2r'(t)\sin(t) + r''\cos(t) \\ -r(t)\sin(t) + 2r'(t)\cos(t) + r''(t)\sin(t) \\ (r'(t)^2 - r''(t))e^{-r(t)} \end{pmatrix}$$

Again

$$g''(t) = \lambda v + \mu w$$

implies $\lambda = -r(t)\cos - 2r'(t)\sin + r''(t)\cos$ and $\mu = -r(t)\sin + 2r'(t)\cos + r''(t)\sin$. Thus we have to check whether

$$-\lambda\frac{x}{r}e^{-r} - 2\mu\frac{y}{r}e^{-r} \stackrel{?}{=} (2r'^2 - r'')e^{-r} .$$

Recall that $x = r\cos$ and hence

$$-\lambda x = -r\cos(-r\cos - 2r'\sin + r''\cos) = r^2\cos^2 + 2rr'\cos\sin - rr''\cos^2 .$$

Similarly $y = r\sin$ and

$$-\mu x = -r\sin(-r\sin + 2r'\cos + r''\sin) = r^2\sin^2 - 2rr'\cos\sin - rr''\cos^2 .$$

This gives

$$-\lambda\frac{x}{r} - \mu\frac{y}{r} \;=\; r - r''\,.$$

Therefore

$$-\lambda\frac{x}{r}e^{-r} - 2\mu\frac{y}{r}e^{-r} \;=\; (r - r'')e^{-r}\,.$$

hence $r = (r')^2$ implies that the second derivative is also in the tangent plane. As mentioned above $r(t) = \frac{t^2}{4}$ satisfies $r'(t) = \frac{t}{2}$ and $r'(t)^2 \;=\; r(t)$. So that for this very particular choice we can force the second derivative to be in the tangent plane. It is probably clear that this is rather a matter of luck, and in general we should not expect any relation between the tangent plane and the second derivative. At any rate the second derivative in the tangent plane means that the particle is driven by acceleration to a point which lies in the tangent plane. For example imagine that you are in a roller coster moving on a surface and you magnet is pulling your cart and the magnet is always on your left hand side and in this direction the surface does not going up or down. That would be an example of such a scenario.

Alternative solution: You can also use the normal vector

$$n \;=\; \begin{pmatrix} -\frac{\partial z}{\partial x} \\ -\frac{\partial z}{\partial y} \\ 1 \end{pmatrix}$$

and then check the condition

$$n \cdot g''(t) \;\overset{?}{=}\; 0\,.$$

In case one we have $z = \sqrt{1 - x^2 - y^2}$ and

$$n \;=\; \begin{pmatrix} \frac{x}{\sqrt{1-x^2-y^2}} \\ \frac{y}{\sqrt{1-x^2-y^2}} \\ 1 \end{pmatrix}\,.$$

Then for $x = r\cos(t)$ and $y = r\cos(t)$ we get

$$n \cdot g''(t) \;=\; \frac{1}{\sqrt{1-r^2}} - (x^2 + y^2) \neq 0\,.$$

For ii) we have

$$n \;=\; \begin{pmatrix} \frac{x}{r}e^{-r} \\ \frac{y}{r}e^{-r} \\ 1 \end{pmatrix}$$

and hence $x = r\cos$, $y = r\sin$ and thus

$$\begin{aligned} & n \cdot g''(t) \\ & = e^{-r}(\cos(-r\cos - 2r'\sin + r''\cos) + \sin(-r\sin + 2r'\cos + r''\sin) + (r')^2 - r'') \\ & = e^{-r}(-r + r'' + (r')^2 - r'') \;=\; e^{-r}((r')^2 - r)\,. \end{aligned}$$

Thus $(r')^2 - r = 0$ show that we have a solution.

5) Explain a moving frame (see the book).

No 2) i) and v) Can be use to calculate the second order approximation for

$$f(x,y) = \begin{pmatrix} x^2y \\ y^2x \end{pmatrix}$$

at $(1,1)$. Indeed, we have

$$\begin{aligned} f(x,y) &\approx f(1,1) + f'(1,1)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \cdot Hf(1,1)\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} \\ &\quad \frac{1}{2}\left(\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} \right)\left(\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}, \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 + 2(x-1) + (y-1) \\ 1 + (x-1) + 2(y-1) \end{pmatrix} \\ &\quad + \begin{pmatrix} 2(x-1)^2 + 4(x-1)(y-1) \\ 4(x-1)(y-1) + 2(y-1)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 + 2(x-1) + (y-1) + (x-1)^2 + 2(x-1)(y-1) \\ 1 + (x-1) + 2(y-1) + (y-1)^2 + 2(x-1)(y-1) \end{pmatrix} \end{aligned}$$

Problem: Show that

$$f(x,y) = e^{(x-1)^2} + (y-x-1)^2$$

has a global minumum at $(1,1)$.

Solution: We may use also show that $g(x,y) = e^{x^2} + (x-y)^2$ has a global minumum at $(0,0)$. We have

$$g'(x,y) = (2xe^{x^2} + 2(x-y), -2(x-y))$$

and

$$Hg(x,y) = \begin{pmatrix} 2e^{x^2} + 4x^2e^{x^2} + 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Using Sylvester's theorem we see that $Hg(x,y) \ \geq \ 0$ (positive definite) if $2e^{x^2} + 4x^2e^{x^2} + 2 \ \geq \ 0$ and

$$4e^{x^2} + 8x^2e^{x^2} + 4 - 4 \ \geq \ 0$$

is true for all x, y. This is obviously true and hence the minimum is attained when $g'(x,y) = 0$. This means $x = y$ (second coordinate) and $x = 0$, i.e. $x = y = 0$.

Additional Homework

Solution (Hw4 problem2) We want to calculate the derivative of

$$f(x) \ = \ \int_0^x \sqrt{x - \sin^2(s)}ds \ .$$

Let us introduce $F(x,y) = \int_0^x \sqrt{y - \sin^2(s)}ds$ and $g(x) = \begin{pmatrix} x \\ x \end{pmatrix}$. Then we have

$$f(x) \ = \ F(g(x))$$

and (using $g'(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$)

$$f'(x) \ = \ F'(g(x))g'(x) \ = \ \frac{\partial F}{\partial x}(g(x)) + \frac{\partial F}{\partial y}(g(x)) \ .$$

Note, however, that

$$\frac{\partial F}{\partial x}(x,y) \ = \ \sqrt{y - \sin^2(x)}$$

and

$$\frac{\partial F}{\partial y}(x,y) \ = \ \frac{1}{2}\int_0^x \frac{1}{\sqrt{y - \sin^2(s)}}ds \ .$$

Thus for $x = y = 1$ we find (since cos is positive on $[0, \pi/2]$ that

$$\begin{aligned} f'(x) &= \sqrt{1 - \sin^2(1)} + \int_0^1 \frac{1}{\sqrt{1 - \sin^2(s)}}ds \\ &= \sqrt{1 - \sin^2(1)} + \int_0^1 \frac{1}{\cos(s)}ds \ . \end{aligned}$$

For the second integral we use $u = \sin(s)$ and $du = \cos(s)ds$. This gives

$$\begin{aligned} \int \frac{1}{\cos(s)}ds = \int \frac{1}{\cos^2(s)} \cos ds \ &= \ \int \frac{1}{1 - u^2}du \\ &= \frac{1}{2}\int \frac{1}{1-u} + \frac{1}{1+u}du \ = \ \frac{1}{2}[-\ln(1-u) + \ln(1+u)] \\ &= \frac{1}{2}\ln\frac{1+u}{1-u} \ . \end{aligned}$$

We have $u(0) = 0$ and $u(1) = \sin(1)$, and therefore

$$\int_0^1 \frac{1}{\cos(s)} ds \;=\; \frac{1}{2} \ln \frac{1 + \sin(1)}{1 - \sin(1)} \;.$$

Thus we find

$$f(1.1) \approx f(1) + (\cos(1) + \frac{1}{2} \ln \frac{1 + \sin(1)}{1 - \sin(1)})0.1 \;.$$

It remains to evaluate

$$\int_0^1 \sqrt{1 - \sin^2(s)} ds = \int_0^1 \cos(s) ds \;=\; \sin(1) \;.$$

Solution (Hw2 problem2) We want to find a curve γ such that $\gamma(t) \in C$

$$C \;=\; \{(x, y) : x^2 + y^2 = 1\} \;,$$

and $\gamma'(t) = e^t$. Let $r : \mathbb{R} \to C$ be gven by $r(s) = \begin{pmatrix} \cos(s) \\ \sin(s) \end{pmatrix}$. Let $h : [0, a] \to \mathbb{R}$ be an arbtrary function. Then we may define

$$\gamma(t) \;=\; r(h(t))$$

and obtain (by the chain rule!)

$$\gamma'(t) \;=\; r'(h(t))h'(t) \;.$$

Therefore

$$|\gamma'(t)| \;=\; |r'(h(t))||h'(t)| \;.$$

Thus for $h(t) = e^t$ we find

$$|\gamma'(t)| \;=\; e^t \;.$$

Just note that $\gamma(0) = r(1)$ does not start at $(1, 0)$!.