Practice problems

1. Let $a = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. In the following problems decide wether $\{v, w\}$ is a spanning system (linearly independent). If the answer is calculate the matrix with respect to the new system

   i) $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;

   ii) $v = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$, $w = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ for $0 \leq t \leq 2\pi$.

   iii) $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

2. Find the derivative

   i) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2y \\ y^2x \end{pmatrix}$;

   ii) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2y \\ y^2x \\ x^2y^2 \end{pmatrix}$;

   iii) $f\begin{pmatrix} x \\ y \end{pmatrix} = xy^2 + yx^2$;

   iv) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^2 + 2yx \\ x^2 + 2xy \\ x^2y^2 \end{pmatrix}$;

   v) Let $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix}$. Calculate g^{-1} and differentiate

   $$F\begin{pmatrix} x \\ y \end{pmatrix} = g^{-1}fg$$

   for f as in i).

   vi) $f(x) = \cos(x)^{\sin(x^2)}$.

v) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\ y^2 & 2yx \end{pmatrix}$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be functions such that

$$f(g\begin{pmatrix} r \\ \theta \end{pmatrix}) = \begin{pmatrix} r \\ \theta \end{pmatrix} .$$

Show that $f'(g(r,\theta)) = g'(r,\theta)^{-1}$. Apply this to $g\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$.

4. Let $r(t) = \begin{pmatrix} t \\ t^2 \\ \frac{2\sqrt{2}}{3}t^{3/2} \end{pmatrix}$. Calculate the arclength, dr/ds, d^2r/ds^2 and the curvature.

5. In the following two problems you are given a vector function $g : [a,b] \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(g(t)) = c \,,$$

where c is a constant. This means g takes values in a level set of the surface $S = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \}$. Check whether<u>both</u> dg/dt and d^2g/d^2t lie in the tangent plane.

i) $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2$ and $c = 1$, $g(t) = \begin{pmatrix} r\cos(t) \\ r\sin(t) \\ \sqrt{1-r^2} \end{pmatrix}$ for some $0 \leq r \leq 1$.

ii) $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z - e^{-\sqrt{x^2+y^2}}$, $c = 0$ and $g(t) = \begin{pmatrix} r(t)\cos(t) \\ r(t)\sin(t) \\ e^{-r(t)} \end{pmatrix}$ such that $r = (r')^2$. (This means $r(t) = \frac{t^2}{4}$ will work with $r(0) = 0$, but there are more examples).

iii) Why is there no need to check that the dr/dt is tangent?

6. Explain a moving frame.