Remark

In the previous proof we could have used the notation

$$\det[v, w] = v_1 w_2 - w_2 v_1$$

$$= \det\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

$$\det[v + \lambda w, w] = \det[v, w]$$

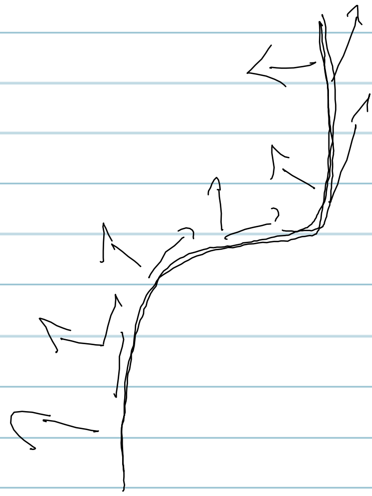
$$\det\left[\frac{d^2 r}{ds^2}, \frac{dr}{ds}\right] = \det\left[\frac{1}{|r'(t)|^2}\frac{d^2 r}{dt^2} + \lambda \frac{dr}{dt}, \frac{1}{|r'(t)|}\frac{dr}{dt}\right]$$

$$= 0$$

$$\boxed{= \frac{1}{|r'(t)|^3} \det\left[\frac{d^2 r}{dt^2}, \frac{dr}{dt}\right]}$$

according!

Geometric interpretation



$r(t) = \begin{pmatrix} t \\ t^3 \end{pmatrix}$

$\boxed{\text{Q In which direction points } \frac{d^2 r}{ds^2}}$

Moving frames

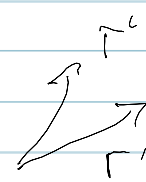
Def ~~Let~~ $r : [a,b] \longrightarrow \mathbb{R}^n$ be a vector function. Then a 2D $\boxed{\text{orthogonal}}$ $\boxed{\text{frame}}$ is given by $v, w : [a,b] \longrightarrow \mathbb{R}^n$ such that

1) For every t we find $\begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}$

st

$$\frac{dr}{dt} = a_{11} v + a_{12} w$$

$$\frac{d^2 r}{dt^2} = a_{21} v + a_{22} w$$



In the span of v, w

2) $v \cdot w = 0$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = (v_1 \ldots v_n) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$= \sum_{j=1}^{n} v_j w_j$$

Def An orthonormal frame is an orthogonal frame satisfy

$$\boxed{|v| = |v(t)| = 1 = |w(t)| = |w|}$$

Pb $r(t) = \begin{pmatrix} t \\ t^3 \end{pmatrix}$ $v \in \mathbb{R}$ $v^\perp = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

An orthogonal frame

$v(t) = \frac{dr}{dt}$ $w(t) = v(t)^\perp$ (always left)

(not normized)

$$\boxed{v(t) = \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix} \qquad w(t) = \begin{pmatrix} -3t^2 \\ 1 \end{pmatrix}}$$

"Better solution" $\quad \frac{dr}{ds} \quad \boxed{\frac{d^2 r}{ds^2}}$

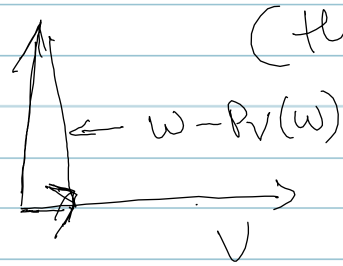
Geometric Definition:

Let $v, w \in \mathbb{R}^2 \ (\mathbb{R}^4)$

$P_v(w)$ = the projection of w onto v

$$= \frac{v \cdot w}{v \cdot v} v$$

(the v component of w)



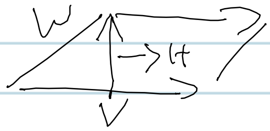
The orthogonal component

$w - P_v(w)$

Note $\quad w = w - P_v(w) + P_v(w)$

$$H = |w - P_v(w)|$$

$\boxed{\text{Fact}}$

Area(

$)^2 = |v|^2|w|^2 - (v \cdot w)^2$

Proof $|w - P_v(w)|^2$ $\qquad P_v(w) = \dfrac{v \cdot w}{v \cdot v} v$

$$= (w - P_v(w)) \cdot (w - P_v(w))$$

$$= w \cdot w - w \cdot P_v(w) - P_v(w) \cdot w + P_v(w) \cdot P_v(w)$$

$$= |w|^2 - 2\frac{v \cdot w}{v \cdot v} v \cdot w + \left(\frac{v \cdot w}{v \cdot v}\right)^2 v \cdot v$$

$$= |w|^2 - \frac{(v \cdot w)^2}{v \cdot v} = |w|^2 - \frac{(v \cdot w)^2}{|v|^2}$$

$$\text{Area}^2 = |v|^2 |w - P_v(w)|^2$$

$$= |v|^2|w|^2 - (v \cdot w)^2 \qquad \square$$

Lemma $s(t) = \int_a^t \left|\frac{dr}{du}\right| du$

$$\frac{d^2s}{dt^2}\frac{ds}{dt} = \frac{d^2r}{dt^2}\cdot\frac{dr}{dt}$$

Proof $\frac{ds}{dt} = \left|\frac{dr}{dt}\right|$ speed

$$\frac{d}{dt}\left|\frac{dr}{dt}\right|^2 = 2\left|\frac{dr}{dt}\right| \frac{d}{dt}\left|\frac{dr}{dt}\right|$$

$$= 2\frac{ds}{dt}\frac{d^2s}{dt^2}$$

$\|$

Product rule

$$\frac{d}{dt}\left(\frac{dr}{dt}\cdot\frac{dr}{dt}\right) = \frac{d^2r}{dt^2}\cdot\frac{dr}{dt} + \frac{dr}{dt}\cdot\frac{d^2r}{dt^2}$$

$$= 2\frac{d^2r}{dt^2}\cdot\frac{dr}{dt} \qquad \square$$

Thm

$$\frac{d^2r}{ds^2} = \frac{\left(\frac{d^2r}{dt^2} - P_{\frac{dr}{dt}}\left(\frac{d^2r}{dt^2}\right)\right)}{\left|\frac{dr}{dt}\right|^2}$$

Proof Recall

$$\frac{d^2r}{ds^2} = \frac{\left|\frac{dr}{dt}\right| \frac{d^2r}{dt^2} - \frac{d}{dt}\left|\frac{dr}{dt}\right| \frac{dr}{dt}}{\left|\frac{dr}{dt}\right|^3}$$

$$\frac{\frac{ds}{dt}}{\left|\frac{ds}{dt}\right|^2}$$

$$= \frac{1}{\left|\frac{dr}{dt}\right|^2}\left[\frac{d^2r}{dt^2} - \frac{d^2s}{dt^2}\,\frac{dt}{ds}\,\frac{dr}{dt}\right]$$

$$= \frac{1}{\left|\frac{dr}{dt}\right|^2}\left[\frac{d^2r}{dt^2} - \frac{\frac{d^2r}{dt^2}\cdot\frac{dr}{dt}}{\frac{dr}{dt}\cdot\frac{dr}{dt}}\,\frac{dr}{dt}\right]$$

$$= \frac{1}{\left|\frac{dr}{dt}\right|^2}\left[P_{\frac{dr}{dt}}\left(\frac{d^2r}{dt^2}\right)\right] \qquad \square$$

Example

$$r(t) = \begin{pmatrix} t \\ t^3 \end{pmatrix} \qquad \frac{dr}{dt} = \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}$$

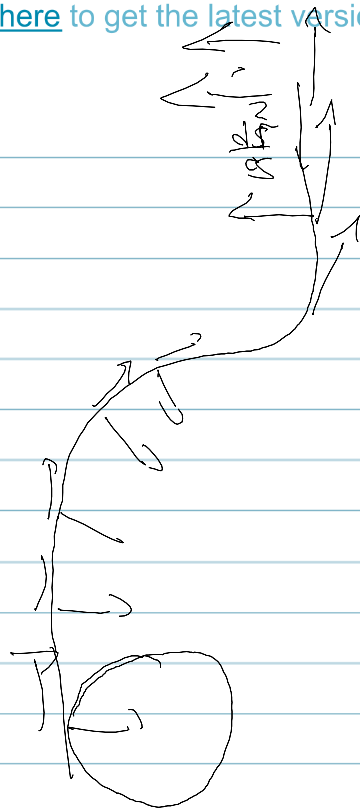
$$\frac{d^2r}{dt^2} = \begin{pmatrix} 0 \\ 6t \end{pmatrix}$$

$$P_{\frac{dr}{dt}}\left(\frac{d^2r}{dt^2}\right) = \begin{pmatrix} 0 \\ 6t \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 6t \end{pmatrix}\begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}\begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}} \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}$$

$$= \frac{1}{1+9t^2}\left[\begin{pmatrix} 0 \\ (1+9t^2)6t \end{pmatrix} - 18t^3 \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix}\right]$$

$$= \frac{1}{1+9t^2}\begin{pmatrix} -18t^3 \\ 6t + 54t^3 - 54t^3 \end{pmatrix}$$

$$= \frac{1}{1+9t^2}\begin{pmatrix} -18t^3 \\ 6t \end{pmatrix} = \frac{6t}{1+9t^2}\begin{pmatrix} -3t^2 \\ 1 \end{pmatrix}$$



points inside

"Better solution"

$\frac{dr}{ds}$ $\frac{d^2r}{ds^2}$