Physical meaning of integral theorems

Maxwells equation's E electric field, B magnetic field

Gauss (1) $\mathrm{div}(E) = \frac{\rho}{\varepsilon_0}$ ρ density of charge

(2) $\mathrm{div}(B) = 0$

Faraday curl(E)
$$\nabla \times E = -\frac{\partial B}{\partial t}$$

current

Ampère curl(B)
$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

we will be concerned with the static laws (1), (2)

$\boxed{(1)}$ let E be an electric field

The flux of E is given by

$$\boxed{\int_S E \cdot dS = \int_S E \cdot n \, dS}$$

$$\text{flux} = \frac{\text{electricity transported through } S}{\text{time}}$$

(rate)

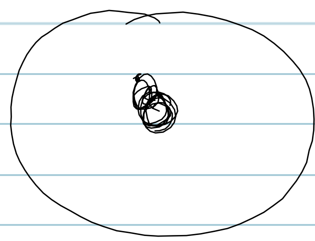
Single particle with charge

Divergence (Gauss) thm

$$\int_{\partial B} E \cdot dS = \int_V \operatorname{div}(E)\, dV$$

for differentiable vector fields E.

<u>Pb</u>



singularity at middle of ball given by a charged particle

particle at origin:

$$E\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{q}{4\pi\varepsilon_0} \frac{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}{\left\| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\|^3}$$

q: charge

$4\pi\varepsilon_0$: constant

div E

$$= \frac{3}{\|x\|^3} - 3\frac{\|x\|^2}{\|x\|^5} = 0$$

<u>lemma</u> let E as above and $r > 0$ then:

$$\int_{\partial B_r(0)} E \cdot n \, dS = \frac{q}{\varepsilon_0} \quad \text{(independent of } r)$$

<u>Cor</u> For any closed body B we have

$$\int_{\partial B} E \cdot dS = \begin{cases} q/\varepsilon_0 & 0 \in B \\ 0 & 0 \notin B \end{cases}$$

Proof $g\binom{\theta}{\varphi} = \begin{pmatrix} r\cos\theta \\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi\end{pmatrix}$

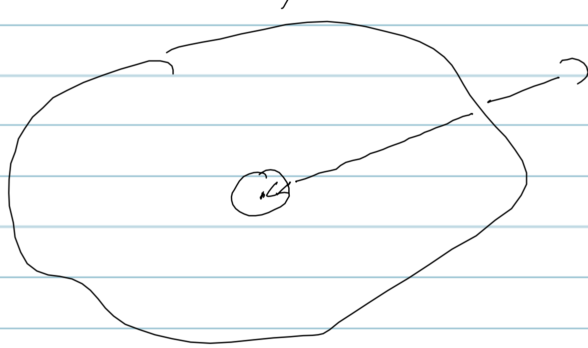
$$N = \frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial\varphi} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi\end{pmatrix} \times \begin{pmatrix} 0 \\ r\sin\theta\,(-\sin\varphi) \\ r\sin\theta\cos\varphi\end{pmatrix}$$

$$= \begin{pmatrix} r^2\cos\theta\sin\theta \\ r^2\sin^2\theta\cos\varphi \\ r^2\sin^2\theta\sin\varphi\end{pmatrix} = r\sin\theta \begin{pmatrix} r\cos\theta\\ r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi\end{pmatrix}$$

$$\int_{\partial B_r} E\cdot n\,dS = \int_0^{2\pi}\int_0^{\pi} \frac{\overbrace{g(\theta,\varphi)\cdot g(\theta,\varphi)}^{=r^2}}{\underbrace{\|g(\theta,\varphi)\|^3}_{=r^3}}\, r\sin\theta\, d\theta\, d\varphi$$

$$= 2\pi \int_0^\pi \sin\theta\,d\theta = 2\pi \left.-\cos\theta\right|_0^{2\pi} = 4\pi \qquad \square$$

The Corollary follows from the divergence theorem



applied to the general annulus

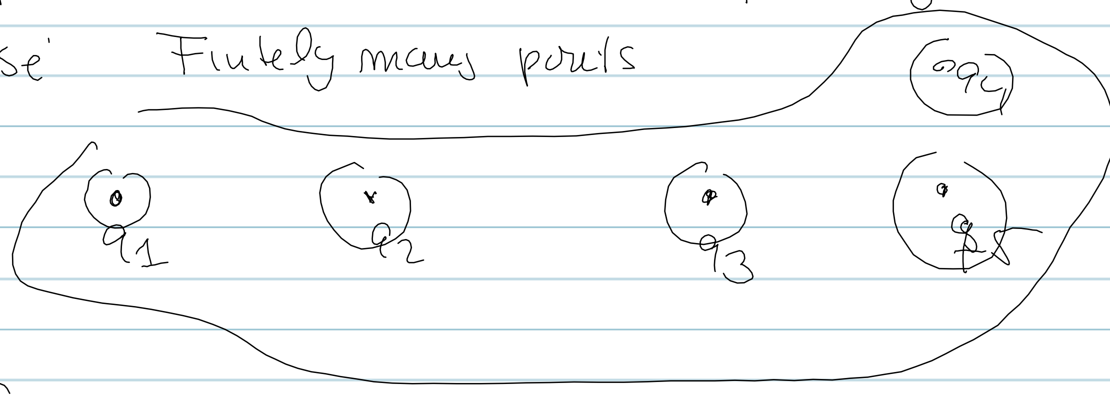
$$\int_{B-B(0,r_0)} E\cdot n\,dS = 0$$

bc $\operatorname{div}(E) = 0$

Many particles

Application: Let ρ = density of charge

Case: Finitely many points



$$\int_{\partial B} E\cdot n\, ds = \sum_{i=1}^{N} \Big(\underbrace{\int_{\partial B - B(0,r)} E\cdot n\, ds}_{=0} + \frac{q_i}{\varepsilon_0} \Big)$$

$$= \int_B \sum_{i=1}^{N} \delta_{a_i}$$

Dirac measure point mass

In ~~general~~ general

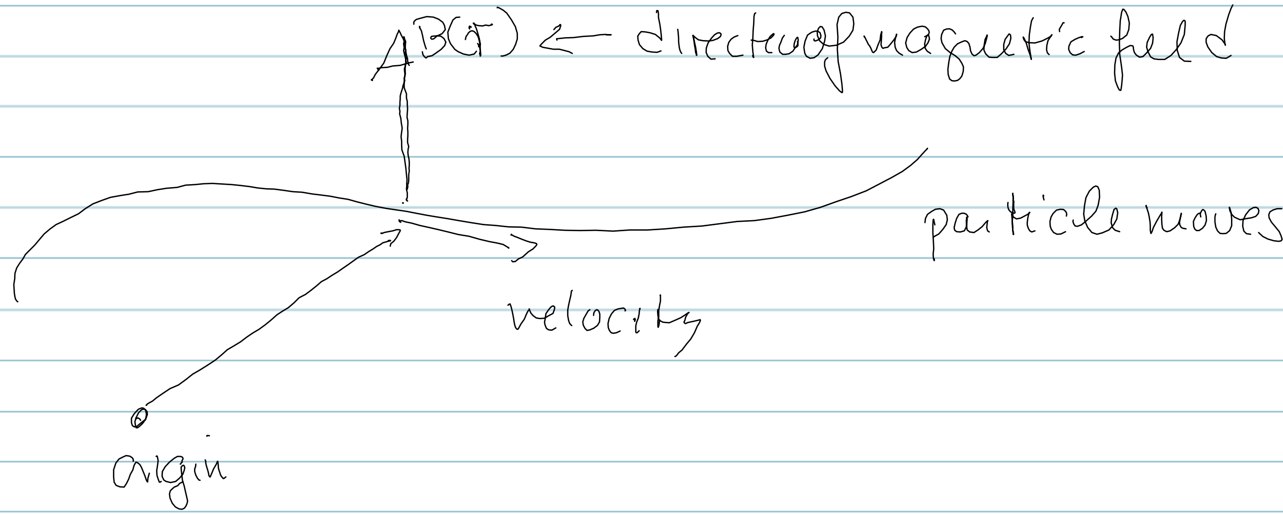
$$\boxed{\int_{\partial B} E\cdot ds = \int_B \rho\, dV}$$

$$\|$$

$$\int \operatorname{div} E\, dV$$

Hence $\boxed{\operatorname{div} E = \frac{\rho}{\varepsilon_0}}$ **Gauss**

$\boxed{\text{No dipoles}}$ r = position



$$B(r) = \text{constant} \; \frac{v \times (r - r_0)}{\|r - r_0\|^3}$$ (located at origin)

<u>Definition</u> $J(x) = v(x)\rho(x)$ velocity × charge density

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

<u>Lemma</u> $\nabla_r \left(\frac{1}{\|r - r_0\|} \right) = - \frac{r - r_0}{\|r - r_0\|^3}$ for fixed r_0

Proof $\nabla \left(\frac{1}{\left((x_1 - r_1)^2 + (x_2 - r_2)^2 - (x_3 - r_3)^2\right)^{1/2}} \right)$ $r_0 = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$

$$= - \frac{1}{\|x - r_0\|^3} \begin{pmatrix} (x_1 - r_1) \\ (x_2 - r_2) \\ (x_3 - r_3) \end{pmatrix}$$

Application $J(x) = v(x)\rho(x)$

$$A(r) = \int_V \frac{J(x)}{\|r-x\|}\, dV(x)$$

$\Longrightarrow$ Then $\boxed{B(r) = -\nabla_r \times A}$

Then

$$\nabla_r \times A = \int_V \nabla_r \times \frac{J(x)}{\|r-x\|}\, dV(x)$$

$$= -\int_V J(x) \times \nabla_r\left(\frac{1}{\|r-x\|}\right) dV(x)$$

$$\nabla \times (fF) = (\nabla \times F)f - F \times \nabla f$$

implies

$$J(x) \times \nabla_r\left(\frac{1}{\|r-x\|}\right) = \underbrace{(\nabla_r \times J(x))}_{=0} \frac{1}{\|r-x\|} - \nabla_r \times \frac{J(x)}{\|r-x\|}$$

$$= -\nabla_r \times \frac{J(x)}{\|r-x\|}$$

$$= -\int J(x) \times \frac{r-x}{\|r-x\|^3}\, dV(x)$$

$= B(r) =$ average magnetic field over V

Short argument: B is the curl of A, $d^2\omega = 0$ implies $\operatorname{div} B = 0$

Cor. $\boxed{\operatorname{div} B = 0}$

Proof $\nabla \times A = d\omega$

$$\omega = A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3$$

(Recall $\int_{\partial D} \varphi \cdot T\, ds = \int_D \operatorname{curl}(\varphi) \cdot dS$)

And $d(d\omega) = 0$ ~~div (P dx~~

~~div (curl) = d~~

$$\Phi = P\, dx_2 \wedge dx_3 + Q\, dx_3 \wedge dx_1 + R\, dx_1 \wedge dx_2$$

$$d\Phi = \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} + \frac{\partial R}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3$$

$\Phi = d\omega \Rightarrow \boxed{d\omega = 0}$ ∎

Remark

The total current wich goes accross a surface is

$$\boxed{I = \int_S J \cdot dS}$$

Ampère's law

J current $\qquad I = \iint_S J \cdot n\, dS$ total current

$= \cancel{\text{den}}$ flux of current

Ampère's law $\qquad \oint_C B \cdot ds = \mu_0 I$

the circulation of the magnetic field gives the total current, up to a constant.

By Stokes theorem

$$\oint_C \overbrace{B \cdot ds}^{\omega} = \mu_0 I = \mu_0 \int_S J \cdot dS$$

$$\| \qquad \cancel{= \mu_0 \int \operatorname{curl}(J)}.$$

$$\int_S \overbrace{\operatorname{curl}(B) \cdot dS}^{d\omega}$$

$\boxed{\text{Be aware of orientation!!}}$

Since S is arbitrary, we get

$$\boxed{\operatorname{Curl}(B) = \mu_0 J}$$

$\boxed{\text{for static flows}}$

Non-Static

The equation of continuity says that

$$\boxed{\mathrm{div}(J) = -\frac{\partial \rho}{\partial t}}$$

Pb $\quad \mathrm{curl}(B) = \mu_0 J$ and $\mathrm{div}(J) = -\frac{\partial \rho}{\partial t}$

do not match

$\mathrm{curl}(B) = \mu_0 J$ means for

$\omega = B_1 dx_1 + B_2 dx_2 + B_3 dx_3$

$d\omega = \cancel{\mathrm{curl}(B)} J_1 \, dx_2 \wedge dx_3 + J_2 \, dx_3 \wedge dx_1 + J_3 \, dx_1 \wedge dx_2$

$\cancel{\mathrm{div}(J) = dd}$

$\mathrm{div}(J) \, d\,\mathrm{Volume} = d(d(\omega)) \, d\,\mathrm{Volume} = 0$

$\neq -\frac{\partial \rho}{\partial t} \mu_0$

(*) later

Modification (Recall $\frac{\partial \rho}{\partial t} = \varepsilon_0 \, \mathrm{div}\left(\frac{\partial E}{\partial t}\right)$)

Hence we replace J by $J + \varepsilon_0 \frac{\partial E}{\partial t}$ and get

$$\boxed{curl(B) = J + \varepsilon_0 \frac{\partial E}{\partial t}}$$ Ampère law

Faradays law

Faraday observed empirically that

$$\boxed{\text{change in magnetic flux} = -\int_C E \cdot d\sigma}$$

$\Phi(t) = \int_S B(t) \cdot dS$ magnetic flux

$\frac{\partial}{\partial t}\Phi(t) = \int_S \frac{\partial}{\partial t} B(t) \cdot dS$ and

$C = \partial S$

$$\int_{\partial S} E \cdot ds = \int_S curl(E) \cdot dS$$

Again by Stokes

Hence $\boxed{curl(E) = -\frac{\partial B}{\partial t}}$